On the Complexity of Probabilistic Abstract Argumentation

Bettina Fazzinga, Sergio Flesca, Francesco Parisi

DIMES Department
University of Calabria
Italy

IJCAI 2013
August 3-9, 2013
Beijing, China

Argumentation in AI

- A general way for representing arguments and relationships (rebuttals) between them
- It allows representing dialogues, making decisions, and handling inconsistency and uncertainty

Abstract Argumentation Framework (AAF) [Dung 1995]: arguments are abstract entities (no attention is paid to their internal structure) that may attack and/or be attacked by other arguments

Example (a simple AAF)

a = Our friends will have great fun at our party on Saturday
b = Saturday will rain (according to the weather forecasting service 1)
c = Saturday will be sunny (according to the weather forecasting service 2)

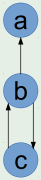
Argumentation in AI

- A general way for representing arguments and relationships (rebuttals) between them
- It allows representing dialogues, making decisions, and handling inconsistency and uncertainty

Abstract Argumentation Framework (AAF) [Dung 1995]: arguments are abstract entities (no attention is paid to their internal structure) that may attack and/or be attacked by other arguments

Example (a simple AAF)

a = Our friends will have great fun at our party on Saturday
b = Saturday will rain (according to the weather forecasting service 1)
c = Saturday will be sunny (according to the weather forecasting service 2)



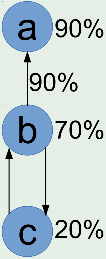
Probabilistic Abstract Argumentation Framework

- Arguments and attacks can be uncertain

Example (modelling uncertainty in our simple AAF)

there is some uncertainty

- about the fact that our friends will have fun at the party
- about the truthfulness of the weather forecasting services
- about the fact that the bad weather forecast actually entails that the party will be disliked by our friends



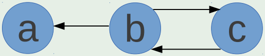
In a **Probabilistic Argumentation Framework (PrAF)** [Li et Al. 2011] both arguments and defeats are associated with probabilities

Semantics for Abstract Argumentations

- In the deterministic setting, several semantics (such as *admissible*, *stable*, *complete*, *grounded*, *preferred*, and *ideal*) have been proposed to identify "reasonable" sets of arguments

Example (AAF)

For instance, $\{a, c\}$ is admissible



- These semantics do make sense in the **probabilistic setting** too: **what is the probability that a set S of arguments is reasonable?** (according to given semantics)

Example (PrAF)

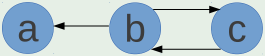
the probability that $\{a, c\}$ is admissible is 0.18

Semantics for Abstract Argumentations

- In the deterministic setting, several semantics (such as *admissible*, *stable*, *complete*, *grounded*, *preferred*, and *ideal*) have been proposed to identify “reasonable” sets of arguments

Example (AAF)

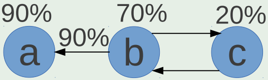
For instance, $\{a, c\}$ is admissible



- These semantics do make sense in the probabilistic setting too: **what is the probability that a set S of arguments is reasonable?** (according to given semantics)

Example (PrAF)

the probability that $\{a, c\}$ is admissible is 0.18



Complexity of Probabilistic Abstract Argumentation

$\text{PROB}^{sem}(S)$ is the problem of *computing the probability* $Pr^{sem}(S)$ *that a set* S *of arguments is reasonable according to semantics sem*

- $\text{PROB}^{sem}(S)$ is the probabilistic counterpart of the problem $\text{VER}^{sem}(S)$ of verifying whether a set S is reasonable according to semantics

sem	$\text{VER}^{sem}(S)$	$\text{PROB}^{sem}(S)$
admissible	*PTIME*	?
stable	*PTIME*	?
complete	*PTIME*	?
grounded	*PTIME*	?
preferred	*coNP*-complete	?
ideal	*coNP*-complete	?

Complexity of Probabilistic Abstract Argumentation

PROB$^{sem}(S)$ is the problem of *computing the probability* $Pr^{sem}(S)$ *that a set* S *of arguments is reasonable according to semantics sem*

- PROB$^{sem}(S)$ is the probabilistic counterpart of the problem VER$^{sem}(S)$ of verifying whether a set S is reasonable according to semantics

sem	VER$^{sem}(S)$	PROB$^{sem}(S)$
admissible	*PTIME*	?
stable	*PTIME*	?
complete	*PTIME*	?
grounded	*PTIME*	?
preferred	*coNP*-complete	?
ideal	*coNP*-complete	?

Complexity of Probabilistic Abstract Argumentation

PROB$^{sem}(S)$ is the problem of *computing the probability* $Pr^{sem}(S)$ *that a set* S *of arguments is reasonable according to semantics sem*

- PROB$^{sem}(S)$ is the probabilistic counterpart of the problem VER$^{sem}(S)$ of verifying whether a set S is reasonable according to semantics

sem	VER$^{sem}(S)$	PROB$^{sem}(S)$
admissible	*PTIME*	*PTIME*
stable	*PTIME*	*PTIME*
complete	*PTIME*	?
grounded	*PTIME*	?
preferred	*coNP*-complete	?
ideal	*coNP*-complete	?

} both tractable (admissible, stable)

Complexity of Probabilistic Abstract Argumentation

PROB$^{sem}(S)$ is the problem of *computing the probability* $Pr^{sem}(S)$ *that a set* S *of arguments is reasonable according to semantics sem*

- PROB$^{sem}(S)$ is the probabilistic counterpart of the problem VER$^{sem}(S)$ of verifying whether a set S is reasonable according to semantics

sem	VER$^{sem}(S)$	PROB$^{sem}(S)$
admissible	*PTIME*	*PTIME*
stable	*PTIME*	*PTIME*
complete	*PTIME*	?
grounded	*PTIME*	?
preferred	*coNP*-complete	$FP^{\#P}$-complete
ideal	*coNP*-complete	$FP^{\#P}$-complete

admissible, stable: both tractable

preferred, ideal: both intractable

Complexity of Probabilistic Abstract Argumentation

$\text{PROB}^{sem}(S)$ is the problem of *computing the probability $Pr^{sem}(S)$ that a set S of arguments is reasonable according to semantics sem*

- $\text{PROB}^{sem}(S)$ is the probabilistic counterpart of the problem $\text{VER}^{sem}(S)$ of verifying whether a set S is reasonable according to semantics

sem	$\text{VER}^{sem}(S)$	$\text{PROB}^{sem}(S)$	
admissible	*PTIME*	*PTIME*	both tractable
stable	*PTIME*	*PTIME*	both tractable
complete	*PTIME*	$FP^{\#P}$-complete	from tractability to intractability
grounded	*PTIME*	$FP^{\#P}$-complete	from tractability to intractability
preferred	*coNP*-complete	$FP^{\#P}$-complete	both intractable
ideal	*coNP*-complete	$FP^{\#P}$-complete	both intractable

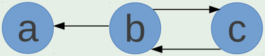
Outline

Basic concepts of Abstract Argumentation

- An *abstract argumentation framework* consists of a set A of *arguments*, and a relation $D \subseteq A \times A$, whose elements are *defeats* (or *attacks*)

Example (AAF)

$A = \{a, b, c\}$
$D = \{\langle b, a\rangle, \langle b, c\rangle, \langle c, b\rangle\}$



- A set $S \subseteq A$ of arguments is *conflict-free* if there are no $a, b \in S$ such that *a defeats b*
- An argument a is *acceptable* w.r.t. $S \subseteq A$ iff $\forall b \in A$ such that *b defeats a*, there is $c \in S$ such that *c defeats b*.

Example (conflict-free and acceptable sets)

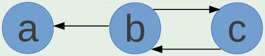
$\{a\}, \{b\}, \{a, c\}$ are conflict-free sets;
a is acceptable w.r.t. $\{c\}$

Basic concepts of Abstract Argumentation

- An *abstract argumentation framework* consists of a set A of *arguments*, and a relation $D \subseteq A \times A$, whose elements are *defeats* (or *attacks*)

Example (AAF)

$A = \{a, b, c\}$
$D = \{\langle b, a\rangle, \langle b, c\rangle, \langle c, b\rangle\}$



- A set $S \subseteq A$ of arguments is *conflict-free* if there are no $a, b \in S$ such that *a defeats b*
- An argument a is *acceptable* w.r.t. $S \subseteq A$ iff $\forall b \in A$ such that *b defeats a*, there is $c \in S$ such that *c defeats b*.

Example (conflict-free and acceptable sets)

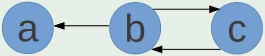
$\{a\}, \{b\}, \{a, c\}$ are conflict-free sets;
a is acceptable w.r.t. $\{c\}$

Semantics for Abstract Argumentation

- Each semantics identifies "reasonable" sets of arguments

semantics ***sem***	**A set** $S \subseteq A$ **of arguments is reasonable according to** ***sem*** **iff**
admissible	S is conflict-free and all its arguments are acceptable w.r.t. S
stable	S is conflict-free and S defeats each argument in $A \setminus S$
complete	S is admissible and S contains all the arguments that are acceptable w.r.t. S
grounded	S is a minimal complete set of arguments
preferred	S is a maximal admissible set of arguments
ideal	S is admissible and S is contained in every preferred set of arguments

Example (semantics for AAF)



admissible sets: $\{a, c\}, \{b\}, \{c\}, \emptyset$
stable sets: $\{a, c\}, \{b\}$
complete sets: $\{a, c\}, \{b\}, \emptyset$
grounded sets: $\emptyset$
preferred sets: $\{a, c\}, \{b\}$
ideal sets: $\emptyset$

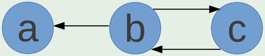
Semantics for Abstract Argumentation

- Each semantics identifies “reasonable” sets of arguments

semantics *sem*	**A set** $S \subseteq A$ **of arguments is reasonable according to** *sem* **iff**
admissible	S is conflict-free and all its arguments are acceptable w.r.t. S
stable	S is conflict-free and S defeats each argument in $A \setminus S$
complete	S is admissible and S contains all the arguments that are acceptable w.r.t. S
grounded	S is a minimal complete set of arguments
preferred	S is a maximal admissible set of arguments
ideal	S is admissible and S is contained in every preferred set of arguments

Example (semantics for AAF)

admissible sets: $\{a, c\}, \{b\}, \{c\}, \emptyset$
stable sets: $\{a, c\}, \{b\}$
complete sets: $\{a, c\}, \{b\}, \emptyset$
grounded sets: $\emptyset$



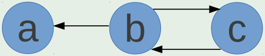
preferred sets: $\{a, c\}, \{b\}$
ideal sets: $\emptyset$

Semantics for Abstract Argumentation

- Each semantics identifies “reasonable” sets of arguments

semantics *sem*	**A set** $S \subseteq A$ **of arguments is reasonable according to** *sem* **iff**
admissible	S is conflict-free and all its arguments are acceptable w.r.t. S
stable	S is conflict-free and S defeats each argument in $A \setminus S$
complete	S is admissible and S contains all the arguments that are acceptable w.r.t. S
grounded	S is a minimal complete set of arguments
preferred	S is a maximal admissible set of arguments
ideal	S is admissible and S is contained in every preferred set of arguments

Example (semantics for AAF)



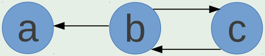
admissible sets: $\{a, c\}, \{b\}, \{c\}, \emptyset$
stable sets: $\{a, c\}, \{b\}$
complete sets: $\{a, c\}, \{b\}, \emptyset$
grounded sets: $\emptyset$
preferred sets: $\{a, c\}, \{b\}$
ideal sets: $\emptyset$

Semantics for Abstract Argumentation

- Each semantics identifies “reasonable” sets of arguments

semantics ***sem***	**A set $S \subseteq A$ of arguments is reasonable according to** ***sem*** **iff**
admissible	S is conflict-free and all its arguments are acceptable w.r.t. S
stable	S is conflict-free and S defeats each argument in $A \setminus S$
complete	S is admissible and S contains all the arguments that are acceptable w.r.t. S
grounded	S is a minimal complete set of arguments
preferred	S is a maximal admissible set of arguments
ideal	S is admissible and S is contained in every preferred set of arguments

Example (semantics for AAF)



admissible sets: $\{a, c\}, \{b\}, \{c\}, \emptyset$
stable sets: $\{a, c\}, \{b\}$
complete sets: $\{a, c\}, \{b\}, \emptyset$
grounded sets: $\emptyset$
preferred sets: $\{a, c\}, \{b\}$
ideal sets: $\emptyset$

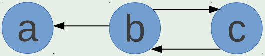
Semantics for Abstract Argumentation

- Each semantics identifies "reasonable" sets of arguments

semantics *sem*	**A set** $S \subseteq A$ **of arguments is reasonable according to** *sem* **iff**
admissible	S is conflict-free and all its arguments are acceptable w.r.t. S
stable	S is conflict-free and S defeats each argument in $A \setminus S$
complete	S is admissible and S contains all the arguments that are acceptable w.r.t. S
grounded	S is a minimal complete set of arguments
preferred	S is a maximal admissible set of arguments
ideal	S is admissible and S is contained in every preferred set of arguments

Example (semantics for AAF)

admissible sets: $\{a, c\}, \{b\}, \{c\}, \emptyset$
stable sets: $\{a, c\}, \{b\}$
complete sets: $\{a, c\}, \{b\}, \emptyset$
grounded sets: $\emptyset$



preferred sets: $\{a, c\}, \{b\}$
ideal sets: $\emptyset$

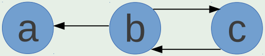
Semantics for Abstract Argumentation

- Each semantics identifies "reasonable" sets of arguments

semantics ***sem***	**A set** $S \subseteq A$ **of arguments is reasonable according to** ***sem*** **iff**
admissible	S is conflict-free and all its arguments are acceptable w.r.t. S
stable	S is conflict-free and S defeats each argument in $A \setminus S$
complete	S is admissible and S contains all the arguments that are acceptable w.r.t. S
grounded	S is a minimal complete set of arguments
preferred	S is a maximal admissible set of arguments
ideal	S is admissible and S is contained in every preferred set of arguments

Example (semantics for AAF)

admissible sets: $\{a, c\}, \{b\}, \{c\}, \emptyset$
stable sets: $\{a, c\}, \{b\}$
complete sets: $\{a, c\}, \{b\}, \emptyset$
grounded sets: $\emptyset$



preferred sets: $\{a, c\}, \{b\}$
ideal sets: $\emptyset$

Basics of Probabilistic Argumentation

- A *PrAF* is a tuple $\langle A, P_A, D, P_D \rangle$ where
  - $\langle A, D \rangle$ is an *AAF*, and
  - P_A and P_D are functions assigning a probability value to each argument in A and defeat in D
- $P_A(a)$ represents the probability that argument a actually occurs
- $P_D(\langle a, b \rangle)$ represents the conditional probability that a defeats b given that both a and b occur

Example (probabilities of arguments and defeats)

$P_A(a) = .9 \quad P_D(\langle b, a \rangle) = .9$
$P_A(b) = .7 \quad P_D(\langle b, c \rangle) = 1$
$P_A(c) = .2 \quad P_D(\langle c, b \rangle) = 1$

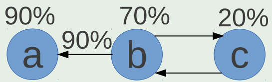
- The issue of how to assign probabilities to arguments/defeats has been investigated in [Hunter 2012, Hunter 2013]

Basics of Probabilistic Argumentation

- A *PrAF* is a tuple $\langle A, P_A, D, P_D \rangle$ where
  - $\langle A, D \rangle$ is an *AAF*, and
  - P_A and P_D are functions assigning a probability value to each argument in A and defeat in D
- $P_A(a)$ represents the probability that argument a actually occurs
- $P_D(\langle a, b \rangle)$ represents the conditional probability that a defeats b given that both a and b occur

Example (probabilities of arguments and defeats)

$$P_A(a) = .9 \qquad P_D(\langle b, a \rangle) = .9$$
$$P_A(b) = .7 \qquad P_D(\langle b, c \rangle) = 1$$
$$P_A(c) = .2 \qquad P_D(\langle c, b \rangle) = 1$$



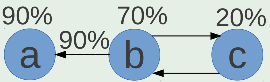
- The issue of how to assign probabilities to arguments/defeats has been investigated in [Hunter 2012, Hunter 2013]

Basics of Probabilistic Argumentation

- A *PrAF* is a tuple $\langle A, P_A, D, P_D\rangle$ where
  - $\langle A, D\rangle$ is an *AAF*, and
  - P_A and P_D are functions assigning a probability value to each argument in A and defeat in D
- $P_A(a)$ represents the probability that argument a actually occurs
- $P_D(\langle a, b\rangle)$ represents the conditional probability that a defeats b given that both a and b occur

Example (probabilities of arguments and defeats)

$$P_A(a) = .9 \qquad P_D(\langle b, a\rangle) = .9$$
$$P_A(b) = .7 \qquad P_D(\langle b, c\rangle) = 1$$
$$P_A(c) = .2 \qquad P_D(\langle c, b\rangle) = 1$$



- The issue of how to assign probabilities to arguments/defeats has been investigated in [Hunter 2012, Hunter 2013]

Meaning of a probabilistic argumentation framework

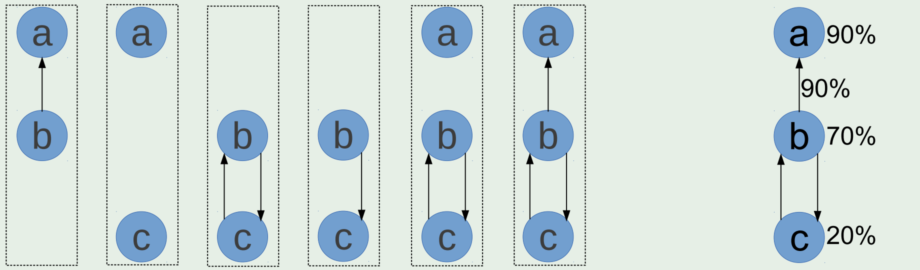
- The meaning of a PrAF is given in terms of possible worlds
- A possible world represents a (deterministic) scenario consisting of some subset of the arguments and defeats of the PrAF
- given a PrAF $\mathcal{F} = \langle A, P_A, D, P_D \rangle$, a possible world w for $\mathcal{F}$ is an AAF $\langle A', D' \rangle$ such that $A' \subseteq A$ and $D' \subseteq D \cap (A' \times A')$.

Example (some possible worlds)

Meaning of a probabilistic argumentation framework

- The meaning of a PrAF is given in terms of possible worlds
- A possible world represents a (deterministic) scenario consisting of some subset of the arguments and defeats of the PrAF
- given a PrAF $\mathcal{F} = \langle A, P_A, D, P_D \rangle$, a possible world w for $\mathcal{F}$ is an AAF $\langle A', D' \rangle$ such that $A' \subseteq A$ and $D' \subseteq D \cap (A' \times A')$.

Example (some possible worlds)



Interpretation for a PrAF

- An interpretation I for a PrAF is a probability distribution over the set of possible worlds
- possible world w is assigned by I the probability $I(w)$ equal to:

$$\prod_{a \in Arg(w)} P_A(a) \times \prod_{a \in A \setminus Arg(w)} (1 - P_A(a)) \times \prod_{\delta \in Def(w)} P_D(\delta) \times \prod_{\delta \in \overline{D}(w) \setminus Def(w)} (1 - P_D(\delta))$$

where $\overline{D}(w) = D \cap (Arg(w) \times Arg(w))$ is the set of defeats that may appear in w

Example (probability of some possible worlds)

$I(w_1) = .9 \times .3 \times .2 = .054$

$I(w_2) = .9 \times .7 \times .2 \times 1 \times 1 \times .1 = .0126$

Interpretation for a PrAF

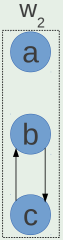
- An interpretation I for a PrAF is a probability distribution over the set of possible worlds
- possible world w is assigned by I the probability $I(w)$ equal to:

$$\prod_{a \in Arg(w)} P_A(a) \times \prod_{a \in A \setminus Arg(w)} (1 - P_A(a)) \times \prod_{\delta \in Def(w)} P_D(\delta) \times \prod_{\delta \in \overline{D}(w) \setminus Def(w)} (1 - P_D(\delta))$$

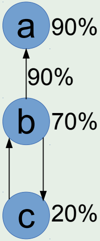
where $\overline{D}(w) = D \cap (Arg(w) \times Arg(w))$ is the set of defeats that may appear in w

Example (probability of some possible worlds)

$I(w_1) = .9 \times .3 \times .2 = .054$



$I(w_2) = .9 \times .7 \times .2 \times 1 \times 1 \times .1 = .0126$



Probability of reasonable sets

- The probability $Pr^{sem}(S)$ that a set S of arguments is reasonable according to a given semantics *sem* is defined as *the sum of the probabilities of the possible worlds w for which S is reasonable according to sem*

Example (probability that $\{a, c\}$ is a admissible set)

In our example, the possible worlds for which $\{a, c\}$ is admissible are:

$$I(w_1) = .054$$
$$I(w_2) = .0126$$
$$I(w_3) = .1134$$

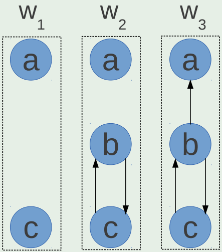
$$Pr^{admissible}(\{a, c\}) = I(w_1) + I(w_2) + I(w_3) = 0.18$$

Probability of reasonable sets

- The probability $Pr^{sem}(S)$ that a set S of arguments is reasonable according to a given semantics *sem* is defined as *the sum of the probabilities of the possible worlds w for which S is reasonable according to sem*

Example (probability that $\{a, c\}$ is a admissible set)

In our example, the possible worlds for which $\{a, c\}$ is admissible are:



$I(w_1) = .054$
$I(w_2) = .0126$
$I(w_3) = .1134$

$$Pr^{admissible}(\{a, c\}) = I(w_1) + I(w_2) + I(w_3) = 0.18$$

Outline

What is the complexity of $\textsc{Prob}^{sem}(S)$?

Definition (Problem $\textsc{Prob}^{sem}(S)$)

Given a PrAF $\langle A, P_A, D, P_D \rangle$, a set $S \subseteq A$ of arguments, and a semantics *sem* in $\{$*admissible, stable, complete, grounded, preferred, ideal*$\}$, $\textsc{Prob}^{sem}(S)$ is the problem of computing the probability $Pr^{sem}(S)$ that the set S is reasonable according to the semantics *sem*

- computing $Pr^{sem}(S)$ by directly applying the definition would require exponential time (it relies on summing the probabilities of an exponential number of possible worlds)
- we shown that $\textsc{Prob}^{sem}(S)$ can be solved in time $O(|S| \cdot |A|)$ for the *admissible* and *stable* semantics
- we shown that $\textsc{Prob}^{sem}(S)$ is $FP^{\#P}$-complete for the *complete, grounded, preferred,* and *ideal* semantics

What is the complexity of $\textsc{Prob}^{sem}(S)$?

Definition (Problem $\textsc{Prob}^{sem}(S)$)

Given a PrAF $\langle A, P_A, D, P_D\rangle$, a set $S \subseteq A$ of arguments, and a semantics *sem* in {*admissible, stable, complete, grounded, preferred, ideal*}, $\textsc{Prob}^{sem}(S)$ is the problem of computing the probability $Pr^{sem}(S)$ that the set S is reasonable according to the semantics *sem*

- computing $Pr^{sem}(S)$ by directly applying the definition would require exponential time (it relies on summing the probabilities of an exponential number of possible worlds)
- we shown that $\textsc{Prob}^{sem}(S)$ can be solved in time $O(|S| \cdot |A|)$ for the *admissible* and *stable* semantics
- we shown that $\textsc{Prob}^{sem}(S)$ is $FP^{\#P}$-complete for the *complete, grounded, preferred,* and *ideal* semantics

What is the complexity of $\textsc{Prob}^{sem}(S)$?

Definition (Problem $\textsc{Prob}^{sem}(S)$)

Given a PrAF $\langle A, P_A, D, P_D \rangle$, a set $S \subseteq A$ of arguments, and a semantics *sem* in {*admissible, stable, complete, grounded, preferred, ideal*}, $\textsc{Prob}^{sem}(S)$ is the problem of computing the probability $Pr^{sem}(S)$ that the set S is reasonable according to the semantics *sem*

- computing $Pr^{sem}(S)$ by directly applying the definition would require exponential time (it relies on summing the probabilities of an exponential number of possible worlds)
- we shown that $\textsc{Prob}^{sem}(S)$ can be solved in time $O(|S| \cdot |A|)$ for the *admissible* and *stable* semantics
- we shown that $\textsc{Prob}^{sem}(S)$ is $FP^{\#P}$-complete for the *complete, grounded, preferred,* and *ideal* semantics

Main idea

- Express the fact that a set S of arguments is admissible [resp., stable] as a probabilistic event $E_{ad}(S)$ [resp., $E_{st}(S)$]
- $Pr^{admissible}(S) = Pr(E_{ad}(S))$ [resp., $Pr^{stable}(S) = Pr(E_{st}(S))$]
- the tractability of PROB$^{admissible}(S)$ [resp. PROB$^{stable}(S)$] follows from the fact that $Pr^{admissible}(S)$ [resp., $Pr^{stable}(S)$)] results in a polynomial-size expression involving only the probabilities of the arguments and the defeats
- this does not hold for the other semantics (*complete, grounded, preferred, and ideal*)

Admissible semantics - probabilistic event

- $E_{ad}(S) = e_1(S) \wedge e_2(S) \wedge e_3(S)$
- $e_1(S)$ is the event that all of the arguments in S occur
- $e_2(S)$ is the event that, given that $e_1(S)$ holds, S is conflict-free
- $e_3(S)$ is the event that, given that $e_1(S)$ holds, for all the arguments d outside S, one of the following events holds:
  - $e_{31}(S, d)$: d does not occur
  - $e_{32}(S, d)$: d occurs and no defeat (d, b), with $b \in S$, occurs
  - $e_{33}(S, d)$: d occurs, there is at least one argument $b \in S$ such that (d, b) occurs, and there is at least one argument $a \in S$ such that (a, d) occurs

Lemma

$Pr^{admissible}(S) = Pr(E_{ad}(S)) = Pr(e_1(S)) \cdot Pr(e_2(S)) \cdot Pr(e_3(S))$

The probabilities of e_1, e_2, and e_3 are as follows (next slides)

Admissible semantics - probabilistic event

- $E_{ad}(S) = e_1(S) \wedge e_2(S) \wedge e_3(S)$
- $e_1(S)$ is the event that all of the arguments in S occur
- $e_2(S)$ is the event that, given that $e_1(S)$ holds, S is conflict-free
- $e_3(S)$ is the event that, given that $e_1(S)$ holds, for all the arguments d outside S, one of the following events holds:
  - $e_{31}(S, d)$: d does not occur
  - $e_{32}(S, d)$: d occurs and no defeat (d, b), with $b \in S$, occurs
  - $e_{33}(S, d)$: d occurs, there is at least one argument $b \in S$ such that (d, b) occurs, and there is at least one argument $a \in S$ such that (a, d) occurs

Lemma

$Pr^{admissible}(S) = Pr(E_{ad}(S)) = Pr(e_1(S)) \cdot Pr(e_2(S)) \cdot Pr(e_3(S))$

The probabilities of e_1, e_2, and e_3 are as follows (next slides)

Admissible semantics - probabilistic event

- $E_{ad}(S) = e_1(S) \wedge e_2(S) \wedge e_3(S)$
- $e_1(S)$ is the event that all of the arguments in S occur
- $e_2(S)$ is the event that, given that $e_1(S)$ holds, S is conflict-free
- $e_3(S)$ is the event that, given that $e_1(S)$ holds, for all the arguments d outside S, one of the following events holds:
  - $e_{31}(S, d)$: d does not occur
  - $e_{32}(S, d)$: d occurs and no defeat (d, b), with $b \in S$, occurs
  - $e_{33}(S, d)$: d occurs, there is at least one argument $b \in S$ such that (d, b) occurs, and there is at least one argument $a \in S$ such that (a, d) occurs

Lemma

$Pr^{admissible}(S) = Pr(E_{ad}(S)) = Pr(e_1(S)) \cdot Pr(e_2(S)) \cdot Pr(e_3(S))$

The probabilities of e_1, e_2, and e_3 are as follows (next slides)

Admissible semantics - probabilistic event

- $E_{ad}(S) = e_1(S) \wedge e_2(S) \wedge e_3(S)$
- $e_1(S)$ is the event that all of the arguments in S occur
- $e_2(S)$ is the event that, given that $e_1(S)$ holds, S is conflict-free
- $e_3(S)$ is the event that, given that $e_1(S)$ holds, for all the arguments d outside S, one of the following events holds:
  - $e_{31}(S, d)$: d does not occur
  - $e_{32}(S, d)$: d occurs and no defeat (d, b), with $b \in S$, occurs
  - $e_{33}(S, d)$: d occurs, there is at least one argument $b \in S$ such that (d, b) occurs, and there is at least one argument $a \in S$ such that (a, d) occurs

Lemma

$$Pr^{admissible}(S) = Pr(E_{ad}(S)) = Pr(e_1(S)) \cdot Pr(e_2(S)) \cdot Pr(e_3(S))$$

The probabilities of e_1, e_2, and e_3 are as follows (next slides)

Admissible semantics - probabilistic event

- $E_{ad}(S) = e_1(S) \wedge e_2(S) \wedge e_3(S)$
- $e_1(S)$ is the event that all of the arguments in S occur
- $e_2(S)$ is the event that, given that $e_1(S)$ holds, S is conflict-free
- $e_3(S)$ is the event that, given that $e_1(S)$ holds, for all the arguments d outside S, one of the following events holds:
  - $e_{31}(S, d)$: d does not occur
  - $e_{32}(S, d)$: d occurs and no defeat (d, b), with $b \in S$, occurs
  - $e_{33}(S, d)$: d occurs, there is at least one argument $b \in S$ such that (d, b) occurs, and there is at least one argument $a \in S$ such that (a, d) occurs

Lemma

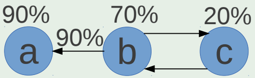
$$Pr^{admissible}(S) = Pr(E_{ad}(S)) = Pr(e_1(S)) \cdot Pr(e_2(S)) \cdot Pr(e_3(S))$$

The probabilities of e_1, e_2, and e_3 are as follows (next slides)

Probability that a set is admissible (1/2)

- $E_{ad}(S) = e_1(S) \wedge e_2(S) \wedge e_3(S)$
- $e_1(S)$ is the event that all of the arguments in S occur
- $Pr(e_1(S)) = \prod_{a \in S} P_A(a)$
- $e_2(S)$ is the event that, given that $e_1(S)$ holds, S is conflict-free
- $Pr(e_2(S)) = \prod_{\substack{\langle a,b\rangle \in D \\ \wedge\, a \in S \wedge\, b \in S}} (1 - P_D(\langle a,b\rangle))$

Example (probability that $\{a, c\}$ is admissible (to be continued))

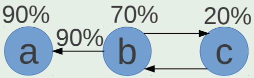


$$Pr^{admissible}(\{a,c\}) = \underbrace{P_A(a) \cdot P_A(c)}_{Pr(e_1(\{a,c\}))} \cdot \underbrace{1}_{Pr(e_2(\{a,c\}))} \cdot Pr(e_3(S))$$

Probability that a set is admissible (1/2)

- $E_{ad}(S) = e_1(S) \wedge e_2(S) \wedge e_3(S)$
- $e_1(S)$ is the event that all of the arguments in S occur
- $Pr(e_1(S)) = \prod_{a \in S} P_A(a)$
- $e_2(S)$ is the event that, given that $e_1(S)$ holds, S is conflict-free
- $Pr(e_2(S)) = \prod_{\substack{\langle a,b\rangle \in D \\ \wedge a \in S \wedge b \in S}} (1 - P_D(\langle a,b\rangle))$

Example (probability that $\{a,c\}$ is admissible (to be continued))



$$Pr^{admissible}(\{a,c\}) = \underbrace{P_A(a) \cdot P_A(c)}_{Pr(e_1(\{a,c\}))} \cdot \underbrace{1}_{Pr(e_2(\{a,c\}))} \cdot Pr(e_3(S))$$

Probability that a set is admissible (2/2)

- $e_3(S)$ is the event that, given that $e_1(S)$ holds, for all the arguments d outside S, one of the following events holds:
  - $e_{31}(S,d)$: d does not occur
  - $e_{32}(S,d)$: d occurs and no defeat (d,b), with $b \in S$, occurs
  - $e_{33}(S,d)$: d occurs, there is at least one argument $b \in S$ such that (d,b) occurs, and there is at least one argument $a \in S$ such that (a,d) occurs
- $Pr(e_3(S)) = \prod_{d \in A \setminus S} \big(Pr(e_{31}(S,d)) + Pr(e_{32}(S,d)) + Pr(e_{33}(S,d)) \big)$ where:
- $Pr(e_{31}(S,d)) = 1 - P_A(d)$
- $Pr(e_{32}(S,d)) = P_A(d) \cdot \prod_{\substack{\langle d,b\rangle \in D \\ \wedge b \in S}} (1 - P_D(\langle d,b\rangle))$
- $Pr(e_{33}(S,d)) = P_A(d) \cdot \Big(1 - \prod_{\substack{\langle d,b\rangle \in D \\ \wedge b \in S}} (1 - P_D(\langle d,b\rangle))\Big) \cdot \Big(1 - \prod_{\substack{\langle a,d\rangle \in D \\ \wedge a \in S}} (1 - P_D(\langle a,d\rangle))\Big)$

Probability that a set is admissible (2/2)

- $e_3(S)$ is the event that, given that $e_1(S)$ holds, for all the arguments d outside S, one of the following events holds:
  - $e_{31}(S,d)$: d does not occur
  - $e_{32}(S,d)$: d occurs and no defeat (d,b), with $b \in S$, occurs
  - $e_{33}(S,d)$: d occurs, there is at least one argument $b \in S$ such that (d,b) occurs, and there is at least one argument $a \in S$ such that (a,d) occurs
- $Pr(e_3(S)) = \prod_{d \in A \setminus S} \big(Pr(e_{31}(S,d)) + Pr(e_{32}(S,d)) + Pr(e_{33}(S,d))\big)$ where:
- $Pr(e_{31}(S,d)) = 1 - P_A(d)$
- $Pr(e_{32}(S,d)) = P_A(d) \cdot \prod_{\substack{\langle d,b\rangle \in D \\ \wedge b \in S}} (1 - P_D(\langle d,b\rangle))$
- $$Pr(e_{33}(S,d)) = P_A(d) \cdot \Big(1 - \prod_{\substack{\langle d,b\rangle \in D \\ \wedge b \in S}} (1 - P_D(\langle d,b\rangle))\Big) \cdot \Big(1 - \prod_{\substack{\langle a,d\rangle \in D \\ \wedge a \in S}} (1 - P_D(\langle a,d\rangle))\Big)$$

Probability that a set is admissible (2/2)

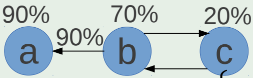
- $e_3(S)$ is the event that, given that $e_1(S)$ holds, for all the arguments d outside S, one of the following events holds:
  - $e_{31}(S,d)$: d does not occur
  - $e_{32}(S,d)$: d occurs and no defeat (d,b), with $b \in S$, occurs
  - $e_{33}(S,d)$: d occurs, there is at least one argument $b \in S$ such that (d,b) occurs, and there is at least one argument $a \in S$ such that (a,d) occurs
- $Pr(e_3(S)) = \prod_{d \in A \setminus S} \big(Pr(e_{31}(S,d)) + Pr(e_{32}(S,d)) + Pr(e_{33}(S,d))\big)$ where:
- $Pr(e_{31}(S,d)) = 1 - P_A(d)$
- $Pr(e_{32}(S,d)) = P_A(d) \cdot \prod_{\substack{\langle d,b\rangle \in D \\ \wedge b \in S}} \big(1 - P_D(\langle d,b\rangle)\big)$
- $Pr(e_{33}(S,d)) = P_A(d) \cdot \Big(1 - \prod_{\substack{\langle d,b\rangle \in D \\ \wedge b \in S}} \big(1 - P_D(\langle d,b\rangle)\big)\Big) \cdot \Big(1 - \prod_{\substack{\langle a,d\rangle \in D \\ \wedge a \in S}} \big(1 - P_D(\langle a,d\rangle)\big)\Big)$

Probability that a set is admissible (2/2)

- $e_3(S)$ is the event that, given that $e_1(S)$ holds, for all the arguments d outside S, one of the following events holds:
  - $e_{31}(S,d)$: d does not occur
  - $e_{32}(S,d)$: d occurs and no defeat (d,b), with $b \in S$, occurs
  - $e_{33}(S,d)$: d occurs, there is at least one argument $b \in S$ such that (d,b) occurs, and there is at least one argument $a \in S$ such that (a,d) occurs
- $Pr(e_3(S)) = \prod_{d \in A \setminus S} \left(Pr(e_{31}(S,d)) + Pr(e_{32}(S,d)) + Pr(e_{33}(S,d))\right)$ where:
- $Pr(e_{31}(S,d)) = 1 - P_A(d)$
- $Pr(e_{32}(S,d)) = P_A(d) \cdot \prod_{\substack{\langle d,b\rangle \in D \\ \wedge b \in S}} (1 - P_D(\langle d,b\rangle))$
- $$Pr(e_{33}(S,d)) = P_A(d) \cdot \left(1 - \prod_{\substack{\langle d,b\rangle \in D \\ \wedge b \in S}} (1 - P_D(\langle d,b\rangle))\right) \cdot \left(1 - \prod_{\substack{\langle a,d\rangle \in D \\ \wedge a \in S}} (1 - P_D(\langle a,d\rangle))\right)$$

Tractability of admissible semantics

Example (probability that $\{a, c\}$ is admissible (continued))



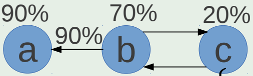
$$
\begin{aligned}
Pr^{admissible}(\{a,c\}) = {} & \underbrace{P_A(a) \cdot P_A(c)}_{Pr(e_1(\{a,c\}))} \cdot \underbrace{1}_{Pr(e_2(\{a,c\}))} \cdot \Big\{ \underbrace{(1 - P_A(b))}_{\color{red}{Pr(e_{31}(\{a,c\},b))}} + \\
& + \underbrace{P_A(b) \cdot (1 - P_D(\langle b,a \rangle)) \cdot (1 - P_D(\langle b,c \rangle))}_{\color{red}{Pr(e_{32}(\{a,c\},b))}} + \\
& + \underbrace{P_A(b) \cdot [1 - (1 - P_D(\langle b,a \rangle))(1 - P_D(\langle b,c \rangle))] \cdot [1 - (1 - P_D(\langle c,b \rangle))]}_{\color{red}{Pr(e_{33}(\{a,c\},b))}} \Big\}
\end{aligned}
$$

Theorem

$\textsc{Prob}^{admissible}(S)$ *can be solved in time* $O(|S| \cdot |A|)$.

Tractability of admissible semantics

Example (probability that $\{a, c\}$ is admissible (continued))



$$Pr^{admissible}(\{a,c\}) = \underbrace{P_A(a)\cdot P_A(c)}_{Pr(e_1(\{a,c\}))} \cdot \underbrace{1}_{Pr(e_2(\{a,c\}))} \cdot \Big\{ \underbrace{(1 - P_A(b)}_{Pr(e_{31}(\{a,c\},b))} +$$

$$+ \underbrace{P_A(b)\cdot(1 - P_D(\langle b,a\rangle))\cdot(1 - P_D(\langle b,c\rangle))}_{Pr(e_{32}(\{a,c\},b))} +$$

$$+ \underbrace{P_A(b)\cdot[1-(1-P_D(\langle b,a\rangle))(1-P_D(\langle b,c\rangle))]\cdot[1-(1-P_D(\langle c,b\rangle))]}_{Pr(e_{33}(\{a,c\},b))} \Big\}$$

Theorem

PROB$^{admissible}(S)$ can be solved in time $O(|S|\cdot|A|)$.

Stable semantics

- probabilistic event that S is stable: $E_{st}(S) = e_1(S) \wedge e_2(S) \wedge e'_3(S)$
- $e'_3(S)$ is the event that, given that $e_1(S)$ holds, for all the arguments d outside S, one of the following events holds:
  - $e_{31}(S, d)$: d does not occur,
  - $e'_{32}(S, d)$: d occurs and it is defeated by S

Lemma

$$Pr^{stable}(S) = Pr(e_1(S)) \cdot Pr(e_2(S)) \cdot$$

$$\cdot \prod_{d \in A \setminus S} \Big\{ \underbrace{1 - P_A(d)}_{Pr(e_{31}(S,d))} + \underbrace{P_A(d) \cdot \Big[1 - \prod_{\langle a, d\rangle \in D \wedge a \in S} (1 - P_D(\langle a, d\rangle))\Big]}_{Pr(e'_{32}(S,d))} \Big\}$$

Theorem

$\text{PROB}^{stable}(S)$ can be solved in time $O(|S| \cdot |A|)$.

Stable semantics

- probabilistic event that S is stable: $E_{st}(S) = e_1(S) \wedge e_2(S) \wedge e'_3(S)$
- $e'_3(S)$ is the event that, given that $e_1(S)$ holds, for all the arguments d outside S, one of the following events holds:
  - $e_{31}(S, d)$: d does not occur,
  - $e'_{32}(S, d)$: d occurs and it is defeated by S

Lemma

$$Pr^{stable}(S) = Pr(e_1(S)) \cdot Pr(e_2(S)) \cdot$$
$$\cdot \prod_{d \in A \setminus S} \Big\{ \underbrace{1 - P_A(d)}_{Pr(e_{31}(S,d))} + \underbrace{P_A(d) \cdot \Big[1 - \prod_{\langle a, d\rangle \in D \wedge a \in S} (1 - P_D(\langle a, d\rangle))\Big]}_{Pr(e'_{32}(S,d))} \Big\}$$

Theorem

PROB$^{stable}(S)$ can be solved in time $O(|S| \cdot |A|)$.

Stable semantics

- probabilistic event that S is stable: $E_{st}(S) = e_1(S) \wedge e_2(S) \wedge e'_3(S)$
- $e'_3(S)$ is the event that, given that $e_1(S)$ holds, for all the arguments d outside S, one of the following events holds:
  - $e_{31}(S, d)$: d does not occur,
  - $e'_{32}(S, d)$: d occurs and it is defeated by S

Lemma

$$Pr^{stable}(S) = Pr(e_1(S)) \cdot Pr(e_2(S)) \cdot$$

$$\cdot \prod_{d \in A \setminus S} \Big\{ \underbrace{1 - P_A(d)}_{Pr(e_{31}(S,d))} + \underbrace{P_A(d) \cdot \big[1 - \prod_{\langle a, d\rangle \in D \wedge a \in S} (1 - P_D(\langle a, d\rangle))\big]}_{Pr(e'_{32}(S,d))} \Big\}$$

Theorem

PROB$^{stable}(S)$ *can be solved in time* $O(|S| \cdot |A|)$.

Complete/Grounded/Preferred/Ideal semantics

Theorem

$\text{PROB}^{complete}(S)$, $\text{PROB}^{grounded}(S)$, $\text{PROB}^{preferred}(S)$ *and* $\text{PROB}^{ideal}(S)$ *are* $FP^{\#P}$*-complete.*

- For complete/grounded semantics:
  - reduction from the $\#P$-hard problem $\#PP2DNF$ (*Partitioned Positive 2DNF*)
  - $\#PP2DNF$ is the problem of counting the number of satisfying assignments of a DNF formula $\phi = C_1 \vee C_2 \vee \cdots \vee C_k$ whose propositional variables are positive and can be partitioned into two sets $X = \{x_1, \ldots, x_n\}$ and $Y = \{y_1, \ldots, y_m\}$, and each clause C_i has the form $x_j \wedge y_\ell$, with $x_j \in X$ and $y_\ell \in Y$
- For preferred/ideal semantics:
  - reduction from $\#P2CNF$ (the problem of counting the number of satisfying assignments of a positive 2CNF formula)
- a function is $FP^{\#P}$-hard iff it is $\#P$-hard

Complete/Grounded/Preferred/Ideal semantics

Theorem

$\text{PROB}^{complete}(S)$, $\text{PROB}^{grounded}(S)$, $\text{PROB}^{preferred}(S)$ and $\text{PROB}^{ideal}(S)$ are $FP^{\#P}$-complete.

- For complete/grounded semantics:
  - reduction from the $\#P$-hard problem $\#PP2DNF$ (*Partitioned Positive 2DNF*)
  - $\#PP2DNF$ is the problem of counting the number of satisfying assignments of a DNF formula $\phi = C_1 \vee C_2 \vee \cdots \vee C_k$ whose propositional variables are positive and can be partitioned into two sets $X = \{x_1, \ldots, x_n\}$ and $Y = \{y_1, \ldots, y_m\}$, and each clause C_i has the form $x_j \wedge y_\ell$, with $x_j \in X$ and $y_\ell \in Y$
- For preferred/ideal semantics:
  - reduction from $\#P2CNF$ (the problem of counting the number of satisfying assignments of a positive 2CNF formula)
- a function is $FP^{\#P}$-hard iff it is $\#P$-hard

Complete/Grounded/Preferred/Ideal semantics

Theorem

$\text{PROB}^{complete}(S)$, $\text{PROB}^{grounded}(S)$, $\text{PROB}^{preferred}(S)$ *and* $\text{PROB}^{ideal}(S)$ *are* $FP^{\#P}$*-complete.*

- For complete/grounded semantics:
  - reduction from the $\#P$-hard problem $\#PP2DNF$ (*Partitioned Positive 2DNF*)
  - $\#PP2DNF$ is the problem of counting the number of satisfying assignments of a DNF formula $\phi = C_1 \vee C_2 \vee \cdots \vee C_k$ whose propositional variables are positive and can be partitioned into two sets $X = \{x_1, \ldots, x_n\}$ and $Y = \{y_1, \ldots, y_m\}$, and each clause C_i has the form $x_j \wedge y_\ell$, with $x_j \in X$ and $y_\ell \in Y$
- For preferred/ideal semantics:
  - reduction from $\#P2CNF$ (the problem of counting the number of satisfying assignments of a positive 2CNF formula)
- a function is $FP^{\#P}$-hard iff it is $\#P$-hard

Outline

Conclusions and future work

- We characterized the complexity of the problem of computing the probability that a set of arguments is reasonable according to a given semantics (admissible/stable/complete/grounded/preferred/ideal)
- for these semantics, the complexity of the problem is either *PTIME* or $FP^{\#P}$-complete
- The fact that the problem is hard for some semantics backs the use of approximate techniques for estimating $Pr^{sem}(S)$ (such as those proposed in [Li et Al. 2011, Fazzinga et Al. 2013])
- Interesting directions for future work are:
  - extending the complexity study to other AAF semantics (such as *semi-stable*, *stage*, *CF2*)
  - characterizing the complexity of the probabilistic version of the *credulous/sceptical acceptance* problem, that is, the problem of computing the probability that an argument belongs to any/every reasonable set according to a given semantics

Conclusions and future work

- We characterized the complexity of the problem of computing the probability that a set of arguments is reasonable according to a given semantics (admissible/stable/complete/grounded/preferred/ideal)
- for these semantics, the complexity of the problem is either *PTIME* or $FP^{\#P}$-complete
- The fact that the problem is hard for some semantics backs the use of approximate techniques for estimating $Pr^{sem}(S)$ (such as those proposed in [Li et Al. 2011, Fazzinga et Al. 2013])
- Interesting directions for future work are:
  - extending the complexity study to other AAF semantics (such as *semi-stable*, *stage*, *CF2*)
  - characterizing the complexity of the probabilistic version of the *credulous/sceptical acceptance* problem, that is, the problem of computing the probability that an argument belongs to any/every reasonable set according to a given semantics

Conclusions and future work

- We characterized the complexity of the problem of computing the probability that a set of arguments is reasonable according to a given semantics (admissible/stable/complete/grounded/preferred/ideal)
- for these semantics, the complexity of the problem is either *PTIME* or $FP^{\#P}$-complete
- The fact that the problem is hard for some semantics backs the use of approximate techniques for estimating $Pr^{sem}(S)$ (such as those proposed in [Li et Al. 2011, Fazzinga et Al. 2013])
- Interesting directions for future work are:
    - extending the complexity study to other AAF semantics (such as *semi-stable*, *stage*, *CF2*)
    - characterizing the complexity of the probabilistic version of the *credulous/sceptical acceptance* problem, that is, the problem of computing the probability that an argument belongs to any/every reasonable set according to a given semantics

Thank you!

... any question?

Selected References

Phan Minh Dung.
On the acceptability of arguments and its fundamental role in nonmonotonic reasoning, logic programming and n-person games.
Artif. Intell., 77(2):321–358, 1995.

Paul E. Dunne and Michael Wooldridge.
Complexity of abstract argumentation.
In *Argumentation in Artificial Intelligence*, 85–104, 2009.

Paul E. Dunne.
The computational complexity of ideal semantics.
Artif. Intell., 173(18):1559–1591, 2009.

Bettina Fazzinga, Sergio Flesca, and Francesco Parisi.
Efficiently Estimating the Probability of Extensions in Abstract Argumentation.
In *SUM*, 106–119, 2013.

Anthony Hunter.
Some foundations for probabilistic abstract argumentation.
In *COMMA*, 117–128, 2012.

Anthony Hunter.
A probabilistic approach to modelling uncertain logical arguments.
Int. J. Approx. Reasoning, 54(1):47–81, 2013.

Hengfei Li, Nir Oren, and Timothy J. Norman.
Probabilistic argumentation frameworks.
In *TAFA*, 1–16, 2011.