

# On strong homogeneity of a class of global optimization algorithms working with infinite and infinitesimal scales

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## Abstract

The necessity to find the global optimum of multiextremal functions arises in many applied problems where finding local solutions is insufficient. One of the desirable properties of global optimization methods is *strong homogeneity* meaning that a method produces the same sequences of points where the objective function is evaluated independently both of multiplication of the function by a scaling constant and of adding a shifting constant. In this paper, several aspects of global optimization using strongly homogeneous methods are considered. First, it is shown that even if a method possesses this property theoretically, numerically very small and large scaling constants can lead to ill-conditioning of the scaled problem. Second, a new class of global optimization problems where the objective function can have not only finite but also infinite or infinitesimal Lipschitz constants is introduced. Third, the strong homogeneity of several Lipschitz global optimization algorithms is studied in the framework of the Infinity Computing paradigm allowing one to work *numerically* with a variety of infinities and infinitesimals. Fourth, it is proved that a class of efficient univariate methods enjoys this property for finite, infinite and infinitesimal scaling and shifting constants. Finally, it is shown that in certain cases the usage of numerical infinities and infinitesimals can avoid ill-conditioning produced by scaling. Numerical experiments illustrating theoretical results are described.

**Keywords:** Lipschitz global optimization, strongly homogeneous methods, numerical infinities and infinitesimals, ill-conditioned problems

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## 1. Introduction

In many applied problems it is required to find the global optimum (minimization problems are considered here, i.e., we talk about the global minimum) of multiextremal non-differentiable functions. Due to the presence of multiple local minima and non-differentiability of the objective function, classical local optimization techniques cannot be used for solving these problems and global optimization methods should be developed (see, e.g., [8, 14, 18, 19, 30, 35, 36, 37, 39, 41]).

One of the desirable properties of global optimization methods (see [7, 35, 41]) is their *strong homogeneity* meaning that a method produces the same sequences of trial points (i.e., points where the objective function  $f(x)$  is evaluated) independently of both shifting  $f(x)$  vertically and its multiplication by a scaling constant. In other words, it can be useful to optimize a scaled function

$$g(x) = g(x; \alpha, \beta) = \alpha f(x) + \beta, \quad \alpha > 0, \quad (1)$$

instead of the original objective function  $f(x)$ . The concept of strong homogeneity has been introduced in [41] where it has been shown that both the P-algorithm (see [40]) and the one-step Bayesian algorithm (see [17]) are strongly homogeneous. The case  $\alpha = 1, \beta \neq 0$  was considered in [7, 35] where a number of methods enjoying this property and called *homogeneous* were studied. It should be mentioned that there exist global optimization methods that are homogeneous or strongly homogeneous and algorithms (see, for instance, the DIRECT algorithm from [14] and a huge number of its modifications) that do not possess this property.

All the methods mentioned above have been developed to work with Lipschitz global optimization problems that can be met very frequently in practical applications (see, e.g., [18, 19, 30, 35, 36, 39]). These methods belong to the class of “Divide-the-best” algorithms introduced in [23]. Efficient methods from this class that iteratively subdivide the search region and estimate local and global Lipschitz constants during the search are studied in this paper, as well. Two kinds of algorithms are taken into consideration: geometric and information ones (see [30, 34, 35, 36]). The first class of algorithms is based on a geometrical interpretation of the Lipschitz condition and takes its origins in the method proposed in [20] that builds a piece-wise linear minorant for the objective function using the Lipschitz condition. The second approach uses a stochastic model developed in [35] that allows one to calculate probabilities of locating global minimizers within each of the subintervals of the search region and is based on the information-statistical algorithm proposed in [35] (for other rich ideas in stochastic global optimization

see [39, 41]). Both classes of methods use in their work different strategies to estimate global and local Lipschitz constants (see, e.g., [20, 30, 32, 34, 35, 36]).

In this paper, it will be shown that several fast univariate methods using local tuning techniques to accelerate the search through a smart balancing of the global and local information collected during the search (see recent surveys in [30, 32]) enjoy the property of the strong homogeneity. In particular, it will be proved that this property is valid for the considered methods not only for finite values of the constants  $\alpha$  and  $\beta$  but for infinite and infinitesimal ones, as well. To prove this result, a new class of global optimization problems with the objective function having infinite or infinitesimal Lipschitz constants is introduced. Numerical computations with functions that can assume infinite and infinitesimal values are executed using the Infinity Computing paradigm allowing one to work *numerically* with a variety of infinities and infinitesimals on a patented in Europe and USA new supercomputer called the Infinity Computer (see, e.g., surveys [24, 28]). This computational methodology has already been successfully applied in optimization and numerical differentiation [3, 5, 6, 26] and in a number of other theoretical and applied research areas such as, e.g., cellular automata [4], hyperbolic geometry [16], percolation [13], fractals [2, 27], infinite series [25, 38], Turing machines [22], numerical solution of ordinary differential equations [1, 33], etc. In particular, in the recent paper [9], numerical infinities and infinitesimals from [24, 28] have been successfully used to handle ill-conditioning in a multidimensional optimization problem.

The importance to have the possibility to work with infinite and infinitesimal scaling/shifting constants  $\alpha$  and  $\beta$  has an additional value due to the following fact. It can happen that even if a method possesses the strong homogeneity property theoretically and the original objective function  $f(x)$  is well-conditioned, numerically very small and/or large finite constants  $\alpha$  and  $\beta$  can lead to the ill-conditioning of the global optimization problem involving  $g(x)$  due to overflow and underflow taking place when  $g(x)$  is constructed from  $f(x)$ . Thus, global minimizers can change their locations and the values of global minima can change, as well. As a result, applying methods possessing the strong homogeneity property to solve these problems will lead to finding the changed values of minima related to  $g(x)$  and not the desired global solution of the original function  $f(x)$  we are interested in. In this paper, it is shown that numerical infinities and infinitesimals and the Infinity Computing framework can help in this situation.

The rest of paper is structured as follows. Section 2 states the problem formally, discusses ill-conditioning induced by scaling, and briefly describes the Infinity Computer framework. It is stressed that the introduction of numerical infini-

ties and infinitesimals allows us to consider a new class of functions having infinite or infinitesimal Lipschitz constants. Section 3 presents geometric and information Lipschitz global optimization algorithms studied in this paper and shows how an adaptive estimation of global and local Lipschitz constants can be performed. So far, the fact whether these methods are strongly homogeneous or not was an open problem even for finite constants  $\alpha$  and  $\beta$ . Section 4 proves that these methods enjoy the strong homogeneity property for finite, infinite, and infinitesimal scaling and shifting constants. Section 5 shows that in certain cases the usage of numerical infinities and infinitesimals can avoid ill-conditioning produced by scaling and illustrates these results numerically. Finally, Section 6 contains a brief conclusion.

## 2. Problem statement, ill-conditioning induced by scaling, and the Infinity Computer framework

### 2.1. Lipschitz global optimization and strong homogeneity

Let us consider the following univariate global optimization problem where it is required to find the global minimum  $f^*$  and global minimizers  $x^*$  such that

$$f^* = f(x^*) = \min_{x \in D} f(x), \quad x \in D = [a, b] \subset \mathbb{R}. \quad (2)$$

It is supposed that the objective function  $f(x)$  can be multiextremal and non-differentiable. Moreover, the objective function  $f(x)$  is supposed to be Lipschitz continuous over the interval  $D$ , i.e.,  $f(x)$  satisfies the following condition

$$|f(x_1) - f(x_2)| \leq L|x_1 - x_2|, \quad x_1, x_2 \in D, \quad (3)$$

where  $L$  is the Lipschitz constant,  $0 < L < \infty$ .

A vast literature is dedicated to the problem (2), (3) and algorithms for its solving (see, e.g., [8, 10, 11, 15, 18, 29, 30, 31, 34, 36, 39]). In practice it can be useful to optimize a scaled function  $g(x)$  from (1) instead of the original objective function  $f(x)$  (see, e.g., [7, 35, 41]). For this kind of problems, the concept of strong homogeneity for global optimization algorithms has been introduced in [41]: An algorithm is called *strongly homogeneous* if it generates the same sequences of trials (evaluations of the objective function) during optimizing the original objective function  $f(x)$  and the scaled function  $g(x)$  from (1), where  $\alpha > 0$  and  $\beta$  are constants (notice that homogeneous methods corresponding to the case  $\alpha = 1, \beta \neq 0$  have been considered originally in [7, 35]). Unfortunately, in practice it is not always possible to obtain correct values of  $g(x)$  for huge and small values of  $\alpha > 0$  and  $\beta$  due to overflows and underflows present if traditional computers and numeral systems are used for evaluation of  $g(x)$  even if the original function  $f(x)$  is well-conditioned.

## 2.2. Ill-conditioning produced by scaling

As an illustration, let us consider the following test problem from [12] shown in Fig. 1.a:

$$f_3(x) = \sum_{k=1}^5 -k \cdot \sin[(k+1)x + k], \quad x \in D = [-10, 10]. \quad (4)$$

The function  $f_3(x)$  has been chosen from the set of 20 test functions described in [12] because it has the highest number of local minima among these functions and the following three global minimizers

$$x_1^* = -0.491, \quad x_2^* = -6.775, \quad x_3^* = 5.792 \quad (5)$$

corresponding to the global minimum

$$f^* = f(x_1^*) = f(x_2^*) = f(x_3^*) = -12.0312. \quad (6)$$

Let us take  $\alpha = 10^{-17}$  and  $\beta = 1$  obtaining so the following function

$$g_3(x) = 10^{-17} f_3(x) + 1. \quad (7)$$

It can be seen from Fig. 1.a and Fig. 1.b that  $f_3(x)$  and  $g_3(x)$  are completely different. If we wish to reestablish  $f_3(x)$  from  $g_3(x)$ , i.e., to compute the inverted scaled function  $\widehat{f}_3(x) = 10^{17}(g_3(x) - 1)$ , then it will not coincide with  $f_3(x)$ . Fig. 1.c shows  $\widehat{f}_3(x)$  constructed from  $g_3(x)$  using MATLAB<sup>®</sup> and the piece-wise linear approximations with the integration step  $h = 0.0001$ .

Thus, this scaling leads to an ill-conditioning. Due to underflows taking place in commonly used numeral systems (in this case, the type *double* in MATLAB<sup>®</sup>), the function  $g_3(x)$  degenerates over many intervals in constant functions and many local minimizers disappear (see Fig. 1.b). In the same time, due to overflows, several local minimizers become global minimizers of the scaled function  $g_3(x)$ . In particular, using the following two commands in MATLAB<sup>®</sup>

$$[gmin, imin] = \min(y), \quad xmin = x(imin)$$

we can calculate an approximation of the global minimum for  $g_3(x)$ . Using the array  $y$  containing the values of  $g_3(x)$  calculated with the stepsize  $h = 0.0001$ , i.e.,

$$y_i = 10^{-17} f_3(x_i) + 1, \quad x_i = -10 + h \cdot (i - 1), \quad i \geq 1,$$

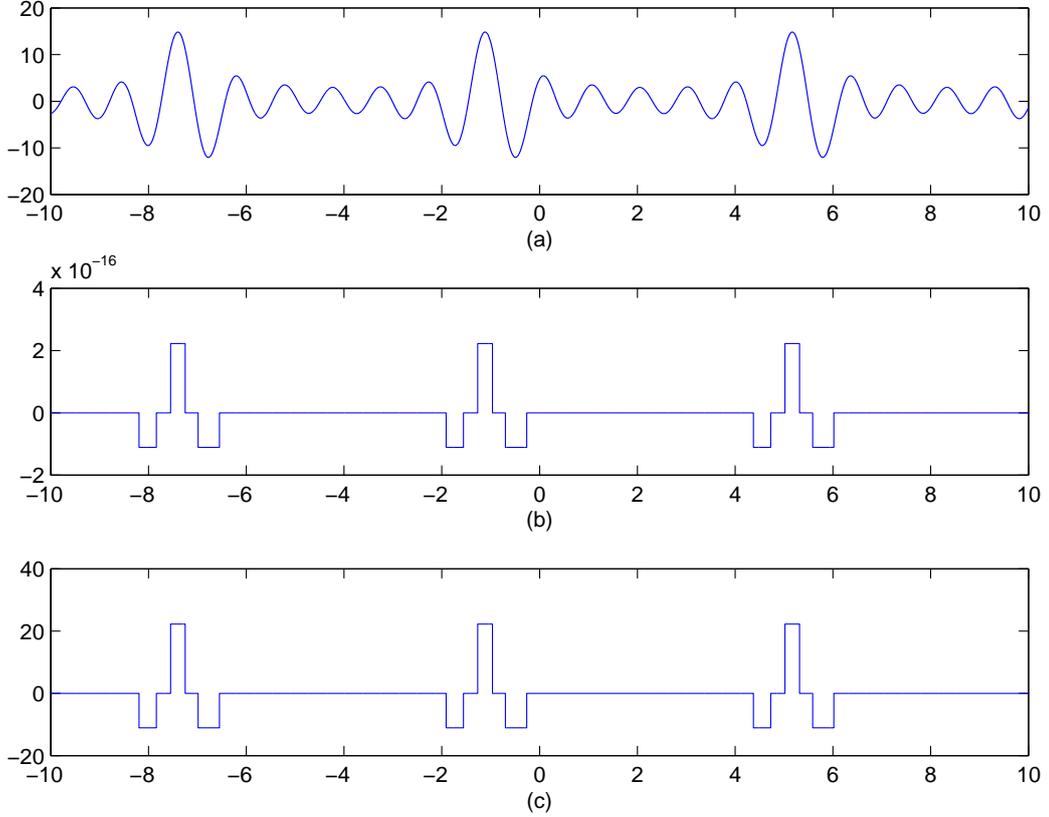


Figure 1: Graphs of (a) the test function (4), (b) the scaled function  $g_3(x)$  from (7) in the logarithmic form, (c) the inverted scaled function  $\hat{f}_3(x) = 10^{17}(g_3(x) - 1)$ . It can be seen that the form of the functions  $g_3(x)$  and  $\hat{f}_3(x)$  are qualitatively different with respect to the original function  $f_3(x)$  due to overflows and underflows.

we get  $(x_{min}, g_{min}) = (-8.194, 1.0)$  being an approximation of the global minimum  $(x^*, g_3(x^*))$  of  $g_3(x)$  that does not coincide with the global minima (5), (6) of the original function  $f_3(x)$ . Thus, due to underflows and overflows, the “wrong” global minimum of the scaled function  $g_3(x)$  has been found. Analogously, due to the same reasons, the inverted function  $\hat{f}_3(x) = 10^{17}(g_3(x) - 1)$  has also different global minima with respect to the original function  $f_3(x)$  (see Fig. 1.c). Clearly, a similar situation can be observed if larger values of  $\alpha$  and  $\beta$  are used (for instance,  $\alpha = 10^{17}$  and  $\beta = 10^{35}$ ).

This example shows that in case of very huge or very small finite values of constants  $\alpha$  and  $\beta$ , even if it has been proved theoretically that a method is strongly

homogeneous, it does not make sense to talk about this property since it is not possible to construct correctly the corresponding scaled functions on the traditional computers.

### 2.3. Infinity Computing briefly

The already mentioned Infinity Computing computational paradigm (see, e.g., surveys in [24, 28]) proposed for working numerically with infinities and infinitesimals can be used in the context of strongly homogeneous global optimization methods, as well. This computational methodology has already been successfully applied in a number of applications mentioned above. In particular, it has been successfully used for studying strong homogeneity of the P-algorithm and the one-step Bayesian algorithm (see [41]) and to handle ill-conditioning in local optimization (see [9]).

In this paper, it is shown that within the Infinity Computing paradigm not only finite, but also infinite and infinitesimal values of  $\alpha$  and  $\beta$  can be adopted. In particular, the ill-conditioning present in the global optimization problem described above in the traditional computational framework can be avoided in certain cases within the Infinity Computing paradigm. This is done by using numerical infinite and/or infinitesimal values of  $\alpha$  and  $\beta$  instead of huge or very small scaling/shifting constants. In order to study the strong homogeneity property with infinite and infinitesimal scaling/shifting constants, let us introduce the Infinity Computing paradigm briefly.

Finite, infinite, and infinitesimal numbers in this framework are represented using the positional numeral system with the infinite base  $\mathbb{1}$  (called *grossone*) introduced as the number of elements of the set of natural numbers<sup>1</sup>. In the  $\mathbb{1}$ -based positional system a number  $C$  expressing the quantity

$$C = c_{p_m} \mathbb{1}^{p_m} + c_{p_{m-1}} \mathbb{1}^{p_{m-1}} + \dots + c_{p_1} \mathbb{1}^{p_1} + c_{p_0} \mathbb{1}^{p_0} + c_{p_{-1}} \mathbb{1}^{p_{-1}} + \dots + c_{p_{-k}} \mathbb{1}^{p_{-k}}, \quad (8)$$

is written in the form

$$C = c_{p_m} \mathbb{1}^{p_m} \dots c_{p_1} \mathbb{1}^{p_1} c_{p_0} \mathbb{1}^{p_0} c_{p_{-1}} \mathbb{1}^{p_{-1}} \dots c_{p_{-k}} \mathbb{1}^{p_{-k}}. \quad (9)$$

In (9), all numerals  $c_i$  are not equal to zero (they can be positive or negative). They are finite, written in a traditional numeral system and are called *grossdigits*,

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<sup>1</sup>It should be emphasized that the Infinity Computing approach allows us a full numerical treatment of both infinite and infinitesimal numbers whereas the non-standard analysis (see [21]) has a symbolic character and, therefore, allows symbolic computations only (see a detailed discussion on this topic in [28]).

whereas all numbers  $p_i$ , called *grosspowers*, are sorted in decreasing order with  $p_0 = 0$ :

$$p_m > \dots > p_1 > p_0 > p_{-1} > \dots > p_{-k}. \quad (10)$$

In the  $\mathbb{1}$ -based numeral system, all finite numbers  $n_{finite}$  can be represented using only one grosspower  $p_0 = 0$  and the grossdigit  $c_0 = n_{finite}$  since  $\mathbb{1}^0 = 1$ . The simplest infinite numbers in this numeral system are expressed by numerals having at least one finite grosspower greater than zero. Simple infinitesimals are represented by numerals having only finite negative grosspowers. The simplest number from this group is  $\mathbb{1}^{-1}$  being the inverse element with respect to multiplication for  $\mathbb{1}$ :

$$\frac{1}{\mathbb{1}} \cdot \mathbb{1} = \mathbb{1} \cdot \frac{1}{\mathbb{1}} = 1.$$

It should be mentioned also, that in this framework numbers having a finite part and infinitesimal ones (i.e., in (9) it follows  $c_j = 0$ ,  $j > 0$ ,  $c_0 \neq 0$ , and  $c_i \neq 0$  for at least one  $i < 0$ ) are called *finite*, while the numbers with only one grossdigit  $c_0 \neq 0$  and  $c_i = 0$ ,  $i \neq 0$ , are called *purely finite*. However, hereinafter all *purely finite* numbers will be called *finite* just for simplicity.

#### 2.4. Functions with infinite/infinitesimal Lipschitz constants

The introduction of the Infinity Computer paradigm allows us to consider univariate global optimization problems with the objective function  $g(x)$  from (1) that can assume not only finite values, but also infinite and infinitesimal ones. It is supposed that the original function  $f(x)$  can assume finite values only and it satisfies condition (3) with a finite constant  $L$ . However, since in (1) the scaling/shifting parameters  $\alpha$  and  $\beta$  can be not only finite, but also infinite and infinitesimal and, therefore, to work with  $g(x)$ , the Infinity Computing framework is required. Thus, the following optimization problem is introduced

$$\min g(x) = \min (\alpha f(x) + \beta), \quad x \in D = [a, b] \subset \mathbb{R}, \alpha > 0, \quad (11)$$

where the function  $f(x)$  can be multiextremal, non-differentiable, and Lipschitz continuous with a finite value of the Lipschitz constant  $L$  from (3). In their turn, the values  $\alpha$  and  $\beta$  can be finite, infinite, and infinitesimal numbers representable in the numeral system (9).

The finiteness of the original Lipschitz constant  $L$  from (3) is the essence of the Lipschitz condition allowing people to construct optimization methods for traditional computers. The scaled objective function  $g(x)$  can assume not only finite, but also infinite and infinitesimal values and, therefore, in these cases it is not

Lipschitzian in the traditional sense. However, the Infinity Computer paradigm extends the space of functions that can be treated theoretically and numerically to functions assuming infinite and infinitesimal values. This fact allows us to extend the concept of Lipschitz functions to the cases where the Lipschitz constant can assume infinite/infinitesimal values.

Let us indicate in the rest of the paper by “ $\hat{\phantom{x}}$ ” all the values related to the function  $g(x)$  and without “ $\hat{\phantom{x}}$ ” the values related to the function  $f(x)$ . The following lemma shows a simple but important property of the Lipschitz constant for the objective function  $g(x)$ .

**Lemma 1.** *The Lipschitz constant  $\hat{L}$  of the function  $g(x) = \alpha f(x) + \beta$ , where  $f(x)$  assumes only finite values and has the finite Lipschitz constant  $L$  over the interval  $[a, b]$  and  $\alpha, \alpha > 0$ , and  $\beta$  can be finite, infinite, and infinitesimal, is equal to  $\alpha L$ .*

*Proof.* The following relation can be obtained from the definition of  $g(x)$  and the fact that  $\alpha > 0$

$$|g(x_1) - g(x_2)| = \alpha |f(x_1) - f(x_2)|, \quad x_1, x_2 \in [a, b].$$

Since  $L$  is the Lipschitz constant for  $f(x)$ , then

$$\alpha |f(x_1) - f(x_2)| \leq \alpha L |x_1 - x_2| = \hat{L} |x_1 - x_2|, \quad x_1, x_2 \in [a, b],$$

and this inequality proves the lemma.  $\square$

Thus, the new Lipschitz condition for the function  $g(x)$  from (1) can be written as

$$|g(x_1) - g(x_2)| \leq \alpha L |x_1 - x_2| = \hat{L} |x_1 - x_2|, \quad x_1, x_2 \in D, \quad (12)$$

where the constant  $L$  from (3) is finite and the quantities  $\alpha$  and  $\hat{L}$  can assume infinite and infinitesimal values.

Notice that in the introduced class of functions infinities and infinitesimals are expressed in numerals (9), and Lemma 1 describes the first property of this class. Notice also that symbol  $\infty$  representing a generic infinity cannot be used together with numerals (9) allowing us to distinguish a variety of infinite (and infinitesimal) numbers. Analogously, Roman numerals (I, II, III, V, X, etc.) that do not allow to express zero and negative numbers are not used in the positional numeral systems where new symbols (0, 1, 2, 3, 5, etc.) are used to express numbers.

Some geometric and information global optimization methods (see [19, 20, 30, 32, 34, 35, 36]) used for solving the traditional Lipschitz global optimization problem (2) are adopted hereinafter for solving the problem (11). A general scheme describing these methods is presented in the next section.

### 3. A General Scheme describing geometric and information algorithms

Methods studied in this paper have a similar structure and belong to the class of “Divide-the-best” global optimization algorithms introduced in [23]. They can have the following differences in their computational schemes distinguishing one algorithm from another:

- (i) Methods are either Geometric or Information (see [30, 35, 36] for detailed descriptions of these classes of methods);
- (ii) Methods can use different approaches for estimating the Lipschitz constant: an a priori estimate, a global adaptive estimate, and two local tuning techniques: Maximum Local Tuning (MLT) and Maximum-Additive Local Tuning (MALT) (see [30, 32, 36] for detailed descriptions of these approaches).

The first difference, (i), consists of the choice of characteristics  $R_i$  for the subintervals  $[x_{i-1}, x_i]$ ,  $2 \leq i \leq k$ , where the points  $x_i$ ,  $1 \leq i \leq k$ , are called *trial points* and are points where the objective function  $g(x)$  has been evaluated during previous iterations:

$$R_i = \begin{cases} \frac{z_i + z_{i-1}}{2} - l_i \frac{x_i - x_{i-1}}{2}, & \text{for geometric methods,} \\ 2(z_i + z_{i-1}) - l_i(x_i - x_{i-1}) - \frac{(z_i - z_{i-1})^2}{l_i(x_i - x_{i-1})}, & \text{for information methods,} \end{cases} \quad (13)$$

where  $z_i = g(x_i)$  and  $l_i$  is an estimate of the Lipschitz constant for the subinterval  $[x_{i-1}, x_i]$ ,  $2 \leq i \leq k$ .

The second distinction, (ii), is related to four different strategies used to estimate the Lipschitz constant  $L$ . The first one consists of applying an a priori given estimate  $\bar{L} > L$ . The second way is to use an adaptive global estimate of the Lipschitz constant  $L$  during the search (the word *global* means that the same estimate is used for the whole region  $D$ ). The global adaptive estimate  $\bar{L}_k$  can be calculated as follows

$$\bar{L}_k = \begin{cases} r \cdot H^k, & \text{if } H^k > 0, \\ 1, & \text{otherwise,} \end{cases} \quad (14)$$

where  $r > 0$  is a reliability parameter and

$$H^k = \max\{H_i : 2 \leq i \leq k\}, \quad (15)$$

$$H_i = \frac{|z_i - z_{i-1}|}{x_i - x_{i-1}}, \quad 2 \leq i \leq k. \quad (16)$$

Finally, the Maximum (MLT) and Maximum-Additive (MALT) local tuning techniques consist of estimating local Lipschitz constants  $l_i$  for each subinterval  $[x_{i-1}, x_i]$ ,  $2 \leq i \leq k$ , as follows

$$l_i^{MLT} = \begin{cases} r \cdot \max\{\lambda_i, \gamma_i\}, & \text{if } H^k > 0, \\ 1, & \text{otherwise,} \end{cases} \quad (17)$$

$$l_i^{MALT} = \begin{cases} r \cdot \max\{H_i, \frac{\lambda_i + \gamma_i}{2}\}, & \text{if } H^k > 0, \\ 1, & \text{otherwise,} \end{cases} \quad (18)$$

where  $H_i$  is from (16), and  $\lambda_i$  and  $\gamma_i$  are calculated as follows

$$\lambda_i = \max\{H_{i-1}, H_i, H_{i+1}\}, \quad 2 \leq i \leq k, \quad (19)$$

$$\gamma_i = H^k \frac{(x_i - x_{i-1})}{X^{max}}, \quad (20)$$

with  $H^k$  from (15) and

$$X^{max} = \max\{x_i - x_{i-1} : 2 \leq i \leq k\}. \quad (21)$$

When  $i = 2$  and  $i = k$  only  $H_2, H_3$ , and  $H_{k-1}, H_k$ , should be considered, respectively, in (19).

After these preliminary descriptions we are ready to describe the General Scheme (GS) of algorithms studied in this paper.

**Step 0. Initialization.** Execute first two trials at the points  $a$  and  $b$ , i. e.,  $x^1 := a$ ,  $z^1 := g(a)$  and  $x^2 := b$ ,  $z^2 := g(b)$ . Set the iteration counter  $k := 2$ . Suppose that  $k \geq 2$  iterations of the algorithm have already been executed. The iteration  $k + 1$  consists of the following steps.

**Step 1. Reordering.** Reorder the points  $x^1, \dots, x^k$  (and the corresponding function values  $z^1, \dots, z^k$ ) of previous trials by subscripts so that

$$a = x_1 < \dots < x_k = b, \quad z_i = g(x_i), \quad 1 \leq i \leq k.$$

**Step 2. Estimates of the Lipschitz constant.** Calculate the current estimates  $l_i$  of the Lipschitz constant for each subinterval  $[x_{i-1}, x_i]$ ,  $2 \leq i \leq k$ , in one of the following ways.

**Step 2.1. A priori given estimate.** Take an a priori given estimate  $\bar{L}$  of the Lipschitz constant for the whole interval  $[a, b]$ , i. e., set  $l_i := \bar{L}$ .

**Step 2.2.** *Global estimate.* Set  $l_i := \bar{L}_k$ , where  $\bar{L}_k$  is from (14).

**Step 2.3.** “*Maximum*” *local tuning.* Set  $l_i := l_i^{MLT}$ , where  $l_i^{MLT}$  is from (17).

**Step 2.4.** “*Maximum-Additive*” *local tuning.* Set  $l_i := l_i^{MALT}$ , where  $l_i^{MALT}$  is from (18).

**Step 3.** *Calculation of characteristics.* Compute for each subinterval  $[x_{i-1}, x_i]$ ,  $2 \leq i \leq k$ , its characteristic  $R_i$  by using one of the following rules.

**Step 3.1.** *Geometric methods.*

$$R_i = \frac{z_i + z_{i-1}}{2} - l_i \frac{x_i - x_{i-1}}{2}. \quad (22)$$

**Step 3.2.** *Information methods.*

$$R_i = 2(z_i + z_{i-1}) - l_i(x_i - x_{i-1}) - \frac{(z_i - z_{i-1})^2}{l_i(x_i - x_{i-1})}. \quad (23)$$

**Step 4.** *Interval selection.* Determine an interval  $[x_{t-1}, x_t]$ ,  $t = t(k)$ , for performing the next trial as follows

$$t = \min \arg \min_{2 \leq i \leq k} R_i. \quad (24)$$

**Step 5.** *Stopping rule.* **If**

$$x_t - x_{t-1} \leq \varepsilon, \quad (25)$$

where  $\varepsilon > 0$  is a given accuracy of the global search, **then Stop** and take as an estimate of the global minimum  $g^*$  the value  $g_k^* = \min_{1 \leq i \leq k} \{z_i\}$  obtained at a point  $x_k^* = \arg \min_{1 \leq i \leq k} \{z_i\}$ .

**Otherwise,** go to Step 6.

**Step 6.** *New trial.* Execute the next trial  $z^{k+1} := g(x^{k+1})$  at the point

$$x^{k+1} = \frac{x_t + x_{t-1}}{2} - \frac{z_t - z_{t-1}}{2l_t}. \quad (26)$$

Increase the iteration counter  $k := k + 1$ , and go to Step 1.

#### 4. Strong homogeneity of algorithms belonging to GS for finite, infinite, and infinitesimal scaling/shifting constants

In this section, we study the strong homogeneity of algorithms described in the previous section. This study is executed simultaneously in the traditional and in the Infinity Computing frameworks. In fact, so far, whether these methods were strongly homogeneous or not was an open problem even for finite constants  $\alpha$  and  $\beta$ . In this section, we show that methods belonging to GS enjoy the strong homogeneity property for finite, infinite, and infinitesimal scaling and shifting constants. Recall that all the values related to the function  $g(x)$  are indicated by “ $\widehat{\phantom{x}}$ ” and the values related to the function  $f(x)$  are written without “ $\widehat{\phantom{x}}$ ”.

The following lemma shows how the adaptive estimates of the Lipschitz constant  $\widehat{L}_k$ ,  $\widehat{l}_i^{MLT}$ , and  $\widehat{l}_i^{MALT}$  that can assume finite, infinite, and infinitesimal values are related to the respective original estimates  $\overline{L}_k$ ,  $l_i^{MLT}$ , and  $l_i^{MALT}$  that can be finite only.

**Lemma 2.** *Let us consider the function  $g(x) = \alpha f(x) + \beta$ , where  $f(x)$  assumes only finite values and has a finite Lipschitz constant  $L$  over the interval  $[a, b]$  and  $\alpha$ ,  $\alpha > 0$ , and  $\beta$  can be finite, infinite and infinitesimal numbers. Then, the adaptive estimates  $\widehat{L}_k$ ,  $\widehat{l}_i^{MLT}$  and  $\widehat{l}_i^{MALT}$  from (14), (17) and (18) are equal to  $\alpha\overline{L}_k$ ,  $\alpha l_i^{MLT}$  and  $\alpha l_i^{MALT}$ , respectively, if  $H^k > 0$ , and to 1, otherwise.*

*Proof.* It follows from (16) that

$$\widehat{H}_i = \frac{|\widehat{z}_i - \widehat{z}_{i-1}|}{x_i - x_{i-1}} = \frac{\alpha|z_i - z_{i-1}|}{x_i - x_{i-1}} = \alpha H_i. \quad (27)$$

If  $H^k \neq 0$ , then  $H^k = \max_{2 \leq i \leq k} \frac{|z_i - z_{i-1}|}{x_i - x_{i-1}}$  and  $H^k \geq H_i$ ,  $2 \leq i \leq k$ . Thus, using (27) we obtain  $\alpha H^k \geq \alpha H_i = \widehat{H}_i$ , and, therefore,  $\widehat{H}^k = \alpha H^k$  and from (14) it follows  $\widehat{L}_k = \alpha\overline{L}_k$ . On the other hand, if  $H^k = 0$ , then both estimates for the functions  $g(x)$  and  $f(x)$  are equal to 1 (see (14)).

The same reasoning can be used to show the respective results for the local tuning techniques MLT and MALT (see (17) and (18))

$$\begin{aligned} \widehat{\lambda}_i &= \max\{\widehat{H}_{i-1}, \widehat{H}_i, \widehat{H}_{i+1}\} = \alpha \max\{H_{i-1}, H_i, H_{i+1}\}, \\ \widehat{\gamma}_i &= \widehat{H}^k \frac{x_i - x_{i-1}}{X^{max}} = \alpha H^k \frac{x_i - x_{i-1}}{X^{max}} = \alpha \gamma_i, \end{aligned}$$

$$\widehat{l}_i^{MLT} = \begin{cases} r \cdot \max\{\widehat{\lambda}_i, \widehat{\gamma}_i\}, & \text{if } \widehat{H}^k > 0, \\ 1, & \text{otherwise.} \end{cases}$$

$$\widehat{l}_i^{MALT} = \begin{cases} r \cdot \max\{\widehat{H}_i, \frac{\widehat{\lambda}_i + \widehat{\gamma}_i}{2}\}, & \text{if } \widehat{H}^k > 0, \\ 1, & \text{otherwise.} \end{cases}$$

Therefore, we can conclude that

$$\widehat{l}_i^{\{MLT, MALT\}} = \begin{cases} \alpha_i^{\{MLT, MALT\}}, & \text{if } H^k > 0, \\ 1, & \text{otherwise.} \end{cases} \quad \square$$

**Lemma 3.** *Suppose that characteristics  $\widehat{R}_i$ ,  $2 \leq i \leq k$ , for the scaled objective function  $g(x)$  are equal to an affine transformation of the characteristics  $R_i$  calculated for the original objective function  $f(x)$*

$$\widehat{R}_i = \widehat{\alpha}_k R_i + \widehat{\beta}_k, \quad 2 \leq i \leq k, \quad (28)$$

where scales  $\widehat{\alpha}_k$ ,  $\widehat{\alpha}_k > 0$ , and  $\widehat{\beta}_k$  can be finite, infinite, or infinitesimal and possibly different for different iterations  $k$ . Then, the same interval  $[x_{t-1}, x_t]$ ,  $t = t(k)$ , from (24) is selected at each iteration for the next subdivision during optimizing  $f(x)$  and  $g(x)$ , i.e., it follows  $\widehat{t}(k) = t(k)$ .

*Proof.* Since due to (24)  $t = \arg \min_{2 \leq i \leq k} R_i$ , then  $R_t \leq R_i$  and

$$\widehat{\alpha}_k R_t + \widehat{\beta}_k \leq \widehat{\alpha}_k R_i + \widehat{\beta}_k, \quad 2 \leq i \leq k.$$

That, due to (28), can be re-written as

$$\widehat{R}_t = \min_{2 \leq i \leq k} \widehat{R}_i = \widehat{\alpha}_k R_t + \widehat{\beta}_k.$$

Notice that if there are several values  $j$  such that  $R_j = R_t$ , then (see (24)) we have  $t < j, j \neq t$ , i.e., even in this situation it follows  $\widehat{t}(k) = t(k)$ . This observation concludes the proof.  $\square$

The following Theorem shows that methods belonging to the GS enjoy the strong homogeneity property.

**Theorem 1.** *Algorithms belonging to the GS and applied for solving the problem (11) are strongly homogeneous for finite, infinite, and infinitesimal scales  $\alpha > 0$  and  $\beta$ .*

*Proof.* Two algorithms optimizing functions  $f(x)$  and  $g(x)$  will generate the same sequences of trials if the following conditions hold:

- (i) The same interval  $[x_{t-1}, x_t]$ ,  $t = t(k)$ , from (24) is selected at each iteration for the next subdivision during optimizing functions  $f(x)$  and  $g(x)$ , i.e., it follows  $\widehat{t}(k) = t(k)$ .
- (ii) The next trial at the selected interval  $[x_{t-1}, x_t]$  is performed at the same point during optimizing functions  $f(x)$  and  $g(x)$ , i.e., in (26) it follows  $\widehat{x}^{k+1} = x^{k+1}$ .

In order to prove assertions (i) and (ii), let us consider computational steps of the GS. For both functions,  $f(x)$  and  $g(x)$ , Steps 0 and 1 of the GS work with the same interval  $[a, b]$ , do not depend on the objective function, and, as a result, do not influence (i) and (ii). Step 2 is a preparative one, it is responsible for estimating the Lipschitz constants for all the intervals  $[x_{i-1}, x_i]$ ,  $2 \leq i \leq k$  and was studied in Lemmas 1–2 above. Step 3 calculates characteristics of the intervals and, therefore, is directly related to the assertion (i). In order to prove it, we consider computations of characteristics  $\widehat{R}_i$  for all possible cases of calculating estimates  $l_i$  during Step 2 and show that there always possible to indicate constants  $\widehat{\alpha}_k$  and  $\widehat{\beta}_k$  from Lemma 3.

Lemmas 1 and 2 show that for the a priori given finite Lipschitz constant  $L$  for the function  $f(x)$  (see Step 2.1) it follows  $\widehat{L} = \alpha L$ . For the adaptive estimates of the Lipschitz constants for intervals  $[x_{i-1}, x_i]$ ,  $2 \leq i \leq k$ , (see (14), (17), (18) and Steps 2.2 – 2.4 of the GS) we have  $\widehat{l}_i = \alpha l_i$ , if  $H^k > 0$ , and  $\widehat{l}_i = l_i = 1$ , otherwise (remind that the latter corresponds to the situation  $z_i = z_1$ ,  $1 \leq i \leq k$ ). Since Step 3 includes substeps defining information and geometric methods, then the following four combinations of methods with Lipschitz constant estimates computed at one of the substeps of Step 2 can take place:

- (a) The value  $\widehat{l}_i = \alpha l_i$  and the geometric method is used. From (22) we obtain

$$\widehat{R}_i = \frac{\widehat{z}_{i-1} + \widehat{z}_i}{2} - \widehat{l}_i \frac{x_i - x_{i-1}}{2} = \alpha \left( \frac{z_{i-1} + z_i}{2} - l_i \frac{x_i - x_{i-1}}{2} \right) + \beta = \alpha R_i + \beta.$$

Thus, in this case we have  $\widehat{\alpha}_k = \alpha$  and  $\widehat{\beta}_k = \beta$ .

- (b) The value  $\widehat{l}_i = \alpha l_i$  and the information method is used. From (23) we get

$$\widehat{R}_i = 2(\widehat{z}_i + \widehat{z}_{i-1}) - \widehat{l}_i(x_i - x_{i-1}) - \frac{(\widehat{z}_i - \widehat{z}_{i-1})^2}{\widehat{l}_i(x_i - x_{i-1})} =$$

$$2\alpha(z_i + z_{i-1}) + 4\beta - \alpha l_i(x_i - x_{i-1}) - \frac{\alpha^2(z_i - z_{i-1})^2}{\alpha l_i(x_i - x_{i-1})} = \alpha R_i + 4\beta.$$

Therefore, in this case it follows  $\widehat{\alpha}_k = \alpha$  and  $\widehat{\beta}_k = 4\beta$ .

- (c) The value  $\widehat{l}_i = l_i = 1$  and the geometric method is considered. Since in this case  $z_i = z_1, 1 \leq i \leq k$ , then for the geometric method (see (22)) we have

$$\begin{aligned} \widehat{R}_i &= \frac{\widehat{z}_{i-1} + \widehat{z}_i}{2} - \widehat{l}_i \frac{x_i - x_{i-1}}{2} = \widehat{z}_1 - \frac{x_i - x_{i-1}}{2} = \\ &= \alpha z_1 + \beta - \frac{x_i - x_{i-1}}{2} = R_i + \alpha z_1 - z_1 + \beta. \end{aligned}$$

Thus, in this case we have  $\widehat{\alpha}_k = 1$  and  $\widehat{\beta}_k = z_1(\alpha - 1) + \beta$ .

- (d) The value  $\widehat{l}_i = l_i = 1$  and the information method is used. Then, the characteristics (see (23)) are calculated as follows

$$\widehat{R}_i = 2(\widehat{z}_i + \widehat{z}_{i-1}) - \widehat{l}_i(x_i - x_{i-1}) - \frac{(\widehat{z}_i - \widehat{z}_{i-1})^2}{\widehat{l}_i(x_i - x_{i-1})} =$$

$$4\widehat{z}_1 - (x_i - x_{i-1}) = 4\alpha z_1 + 4\beta - (x_i - x_{i-1}) = R_i + 4\alpha z_1 - 4z_1 + 4\beta.$$

Therefore, in this case it follows  $\widehat{\alpha}_k = 1$  and  $\widehat{\beta}_k = 4(z_1(\alpha - 1) + \beta)$ .

Let us show now that assertion (ii) also holds. Since for both the geometric and the information approaches the the same formula (26) for computing  $x^{k+1}$  is used, we should consider only two cases related to the estimates of the Lipschitz constant:

- (a) If  $\widehat{l}_t = \alpha l_t$ , then it follows

$$\widehat{x}^{k+1} = \frac{x_t + x_{t-1}}{2} - \frac{\widehat{z}_t - \widehat{z}_{t-1}}{2\widehat{l}_t} = \frac{x_t + x_{t-1}}{2} - \frac{\alpha(z_t - z_{t-1})}{2\alpha l_t} = x^{k+1}.$$

- (b) If  $\widehat{l}_t = l_t = 1$ , then  $z_i = z_1, 1 \leq i \leq k$ , and we have

$$\widehat{x}^{k+1} = \frac{x_t + x_{t-1}}{2} - \frac{\widehat{z}_t - \widehat{z}_{t-1}}{2\widehat{l}_t} = \frac{x_t + x_{t-1}}{2} = x^{k+1}.$$

This result concludes the proof.  $\square$

## 5. Numerical illustrations

In order to illustrate the behavior of methods belonging to the GS in the Infinity Computer framework, the following three algorithms being examples of concrete implementations of the GS have been tested:

- **Geom-AL:** Geometric method with an a priori given overestimate of the Lipschitz constant. It is constructed by using Steps 2.1 and 3.1 in the GS.
- **Inf-GL:** Information method with the global estimate of the Lipschitz constant. It is formed by using Steps 2.2 and 3.2 in the GS.
- **Geom-LTM:** Geometric method with the “Maximum” local tuning. It is built by applying Steps 2.3 and 3.1 in the GS.

The algorithm Geom-AL has one parameter – an a priori given overestimate of the Lipschitz constant. In algorithms Geom-LTM and Inf-GL, the Lipschitz constant is estimated during the search and the reliability parameter  $r$  is used. In this work, the values of the Lipschitz constant of the functions  $f(x)$  for the algorithm Geom-AL have been taken from [15] (and multiplied by  $\alpha$  for the function  $g(x)$ ). The values of the parameter  $r$  for the algorithms Geom-LTM and Inf-GL have been set to 1.1 and 1.5, respectively. The value  $\epsilon = 10^{-4}(b - a)$  has been used in the stopping criterion (25).

Recall that (see Section 2) huge or very small scaling/shifting constants can provoke the ill-conditioning of the scaled function  $g(x)$  in the traditional computational framework. In the Infinity Computing framework, the positional numeral system (9) allows us to avoid ill-conditioning and to work safely with infinite and infinitesimal scaling/shifting constants if the respective grossdigits and grosspowers are not too large or too small. In order to illustrate this fact the following two pairs of the values  $\alpha$  and  $\beta$  have been used in our experiments:  $(\alpha_1, \beta_1) = (\mathbb{1}^{-1}, \mathbb{1})$  and  $(\alpha_2, \beta_2) = (\mathbb{1}, \mathbb{1}^2)$ . The corresponding grossdigits and grosspowers involved in their representation are, respectively: 1 and  $-1$  for  $\alpha_1$ ; 1 and 1 for  $\beta_1$ ; 1 and 1 for  $\alpha_2$ ; and 1 and 2 for  $\beta_2$ . It can be seen that all of these constants are numbers that do not provoke instability in numerical operations. Hereinafter scaled functions constructed using constants  $(\alpha_1, \beta_1)$  are indicated as  $g(x)$  and functions using  $(\alpha_2, \beta_2)$  are designated as  $h(x)$ .

The algorithms Geom-AL, Inf-GL, and Geom-LTM have been tested on 20 global optimization problems from [12, 15] and on the respective scaled functions  $g(x)$  and  $h(x)$  constructed from them. It has been obtained that on all 20 test problems with infinite and infinitesimal constants  $(\alpha_1, \beta_1)$  and  $(\alpha_2, \beta_2)$  the results

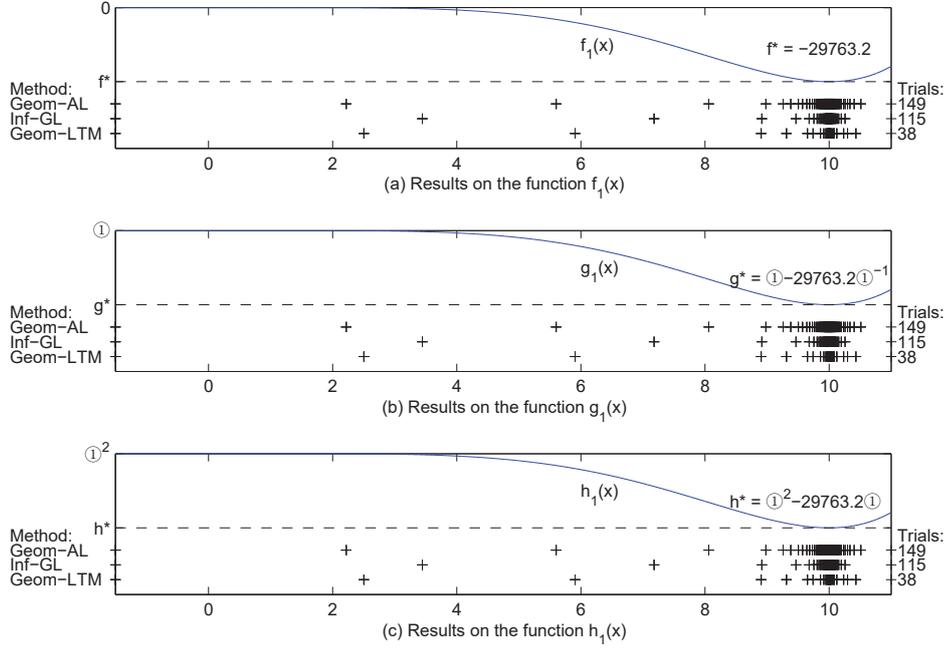


Figure 2: Results for (a) the original test function  $f_1(x)$  from [12, 15], (b) the scaled test function  $g_1(x) = \mathbb{1}^{-1}f_1(x) + \mathbb{1}$ , (c) the scaled test function  $h_1(x) = \mathbb{1}f_1(x) + \mathbb{1}^2$ . Trials are indicated by the signs “+” under the graphs of the functions and the number of trials for each method is indicated on the right. The results coincide for each method on all three test functions.

on the original functions  $f(x)$  from [12, 15] and on scaled functions  $g(x)$  and  $h(x)$  coincide. To illustrate this fact, let us consider the first three problems from the set of 20 tests (see Fig. 2.a, Fig. 3.a, and Fig. 4.a). They are defined as follows

$$f_1(x) = \frac{1}{6}x^6 - \frac{52}{25}x^5 + \frac{39}{80}x^4 + \frac{71}{10}x^3 - \frac{79}{20}x^2 - x + \frac{1}{10},$$

$$f_2(x) = \sin(x) + \sin\frac{10x}{3},$$

$$f_3(x) = \sum_{k=1}^5 -k \cdot \sin[(k+1)x + k].$$

In Fig. 2.b, Fig. 3.b, and Fig. 4.b, the results for the scaled functions

$$g_i(x) = \mathbb{1}^{-1}f_i(x) + \mathbb{1}, \quad i = 1, 2, 3,$$

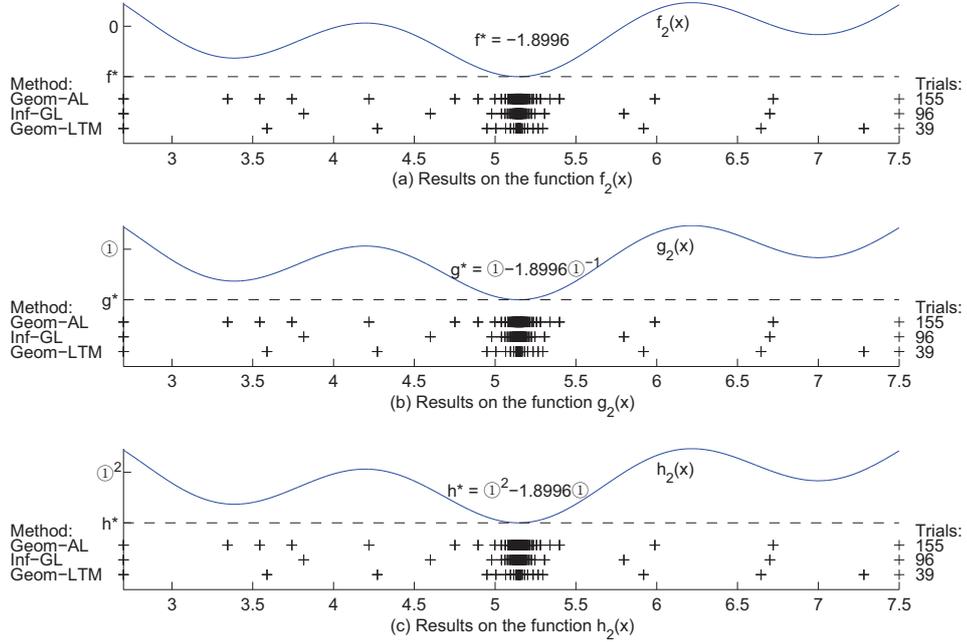


Figure 3: Results for (a) the original test function  $f_2(x)$  from [12, 15], (b) the scaled test function  $g_2(x) = \mathbb{1}^{-1}f_2(x) + \mathbb{1}$ , (c) the scaled test function  $h_2(x) = \mathbb{1}f_2(x) + \mathbb{1}^2$ . Trials are indicated by the signs “+” under the graphs of the functions and the number of trials for each method is indicated on the right. The results coincide for each method on all three test functions.

are presented and in Fig. 2.c, Fig. 3.c, and Fig. 4.c, the results for the scaled functions

$$h_i(x) = \mathbb{1}f_i(x) + \mathbb{1}^2, \quad i = 1, 2, 3,$$

are shown. It can be seen that the results coincide for all three methods on all three test functions  $f_i(x)$ ,  $g_i(x)$ , and  $h_i(x)$ ,  $i = 1, 2, 3$ . Analogous results hold for the remaining test problems from [12, 15].

In particular, it can be seen from these experiments that even if the scaling constants  $\alpha$  and  $\beta$  have a different order (e.g., when  $\alpha$  is infinitesimal and  $\beta$  is infinite) the scaled problems continue to be well-conditioned (cf. discussion on ill-conditioning in the traditional framework with finite scaling/shifting constants, see Fig. 1). This fact suggests that even if finite constants of significantly different orders are required,  $\mathbb{1}$  can also be used to avoid the ill-conditioning by substituting very small constants by  $\mathbb{1}^{-1}$  and very huge constants by  $\mathbb{1}$ . In this case, if, for instance,  $\alpha$  is too small (as, e.g., in (7),  $\alpha = 10^{-17}$ ) and  $\beta$  is too large (as, e.g., in (7),

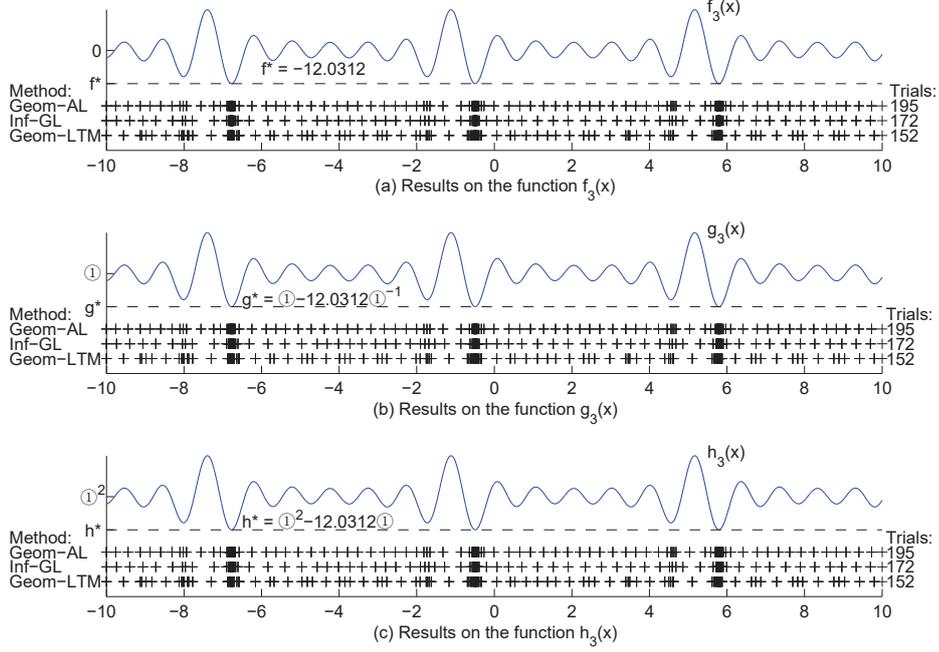


Figure 4: Results for (a) the original test function  $f_3(x)$  from [12, 15], (b) the scaled test function  $g_3(x) = \mathbb{1}^{-1}f_3(x) + \mathbb{1}$ , (c) the scaled test function  $h_3(x) = \mathbb{1}f_3(x) + \mathbb{1}^2$ . The results coincide for each method on all three test functions. The number of trials for each method is indicated on the right.

( $\beta = 1 \gg 10^{-17}$ ), the values  $\alpha_1 = \mathbb{1}^{-1}$  and  $\beta_1 = \mathbb{1}$  can be used in computations instead of  $\alpha = 10^{-17}$  and  $\beta = 1$  avoiding so underflows and overflows. After the conclusion of the optimization process, the global minimum of the original function  $f^*$  can be easily extracted from the solution  $g^* = \alpha_1 f^* + \beta_1 = \mathbb{1}^{-1}f^* + \mathbb{1}$  of the scaled problem using  $\mathbb{1}^{-1}$  and  $\mathbb{1}$  and the original finite constants  $\alpha$  and  $\beta$  can be used to get the required value  $g^* = \alpha f^* + \beta$  (in our case,  $g^* = 10^{-17}f^* + 1$ ).

## 6. Concluding remarks

Univariate Lipschitz global optimization problems have been considered in this paper. Strong homogeneity of global optimization algorithms has been studied in the new computational framework – Infinity Computing. A new class of global optimization problems has been introduced where the objective function can have finite, infinite or infinitesimal Lipschitz constants. The strong homogeneity of a class of geometric and information algorithms used for solving the

univariate Lipschitz global optimization problems belonging to the new class has been proved for finite, infinite, and infinitesimal scaling constants. Numerical experiments executed on a set of test problems taken from the literature confirm the obtained theoretical results.

Moreover, it has been shown that in cases where global optimization problems become ill-conditioned in the traditional computational framework working with finite numbers due to very huge and/or small scaling/shifting constants, applying Infinity Computing can help in certain cases. In this situation it is useful to substitute finite constants provoking problems by infinite and infinitesimal numbers that allow one to avoid ill-conditioning of the scaled problems.

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